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On the Physical Interpretation and the Mathematical Structure of the Combinatorial Hierarchy[1]

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The combinatorial hierarchy model for basic particle processes is based on elementary entities; any representation they may have is discrete and two-valued. We call them *Schnurs* to suggest their most fundamental aspect as concatenating strings. Consider a definite small number of them. Consider an elementary creation act as a result of which two different Schnurs generate a new Schnur which is again different. We speak of this process as a "discrimination." By this process and by this process alone can the complexity of the universe be explored. By concatenations of this process we create more complex entities which are themselves Schnurs at a new level of complexity. Everything plays a dual role in which something comes in from the outside to interact, and also serves as a synopsis or concatenation of such a process. We thus incorporate the observation metaphysic at the start, rejecting Bohr's reduction to the haptic language of common sense and classical physics. Since discriminations

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occur sequentially, our model is consistent with a "fixed past–uncertain future" philosophy of physics. We demonstrate that this model generates four hierarchical levels of rapidly increasing complexity. Concrete interpretation of the four levels of the hierarchy (with cardinals $3, 7, 127, 2^{127}-1 \approx 10^{38}$) associates the three levels which map up and down with the three absolute conservation laws (charge, baryon number, lepton number) and the spin dichotomy. The first level represents $+$, $-$, and $\pm$ unit charge. The second has the quantum numbers of a baryon–antibaryon pair and associated charged meson (e.g., $n\bar{n}, p\bar{n}, p\bar{p}, n\bar{p}, \pi^+, \pi^0, \pi^-$). The third level associates this pair, now including four spin states as well as four charge states, with a neutral lepton–antilepton pair ($e\bar{e}$ or $\nu\bar{\nu}$), each pair in four spin states (total, 64 states)—three charged spinless, three charged spin-1, and a neutral spin-1 mesons (15 states), and a neutral vector boson associated with the leptons; this gives $3+15+3\times15=63$ possible boson states, so a total correct count of $63+64=127$ states. Something like $SU_2 \times SU_3$ and other indications of quark quantum numbers can occur as substructures at the fourth (unstable) level. Breaking into the (Bose) hierarchy by structures with the quantum numbers of a fermion, if this is an electron, allows us to understand Parker-Rhodes' calculation of $m_p/m_e = 1836.1515$ in terms of our interpretation of the hierarchy. A slight extension gives us the usual static approximation to the binding energy of the hydrogen atom, $\alpha^2 m_e c^2$. We also show that the cosmological implications of the theory are in accord with current experience. We conclude that we have made a promising beginning in the physical interpretation of a theory which could eventually encompass all branches of physics.

1. INTRODUCTION: GENERAL PRINCIPLES OF THE COMBINATORIAL HIERARCHY

In this section we are concerned with the basic principles of our combinatorial model of basic physical interactions. This theory was presented at two successive conferences on "Quantum Theory and the Structures of Time and Space" at Tutzing (Bastin, 1976b). We shall compare and contrast our own principles with the central position in those conferences as a convenient and brief way to present the relation of our theory to the basic principles of the quantum theory, since we may regard the central position established at Tutzing as the most coherent existing attempt to establish foundational principles for current quantum theory.

The combinatorial hierarchy model was originally developed (Bastin, 1966) as an attempt to base physics on a single binary process called "discrimination." Sets of "columns" containing only the existence symbols 0, 1 closed under this operation are then viewed as new entities, and the process is repeated. In this way we generate a hierarchy of four levels of rapidly increasing complexity. Although the explicit representation of this hierarchy is not unique, the scheme itself is, as we demonstrate in this paper. Tentative contact with experiment can be made by specific interpretation of the representations, and structural features familiar in the study

of elementary particle physics emerge, including some well-known numerical results. This theory is essentially intended as a conceptual underpinning of the existing formalism of the quantum theory.

The central idea at the Tutzing conferences was a theory of *Ur*'s—basic, discrete, two-valued entities. The claim made by the *Ur* theorists (see particularly von Weizsäcker's 1978 paper) has been that if finitism is firmly and clearly enough embraced, then something very like the usual quantum theoretical formalism can be sustained as a consistent theory and the paradoxes and other perplexities avoided.

A different position has been maintained by Finkelstein (1969, 1977, 1979), who accepts the finitist part of the *Ur* program but considers that further innovation in basic principles is necessary. He adopts a process philosophy, thinking that the elementary discrete constituents of nature must have a principle of concatenation, and that this principle, whatever it may be, must tell us a good deal about the interrelations of the classical and the quantum worlds.

Our theory accords with Finkelstein's demand for innovation beyond the finitist assumption; we adopt the general direction of his "process," or sequential concatenating conjecture. We present a definite model within the class specified by his conjecture, and can claim experimental backing for our model. Our model is distinct from quantum mechanics; it might become equivalent to the latter under special conditions. Some results that would normally be thought to be dependent upon quantum mechanics as a complete theory appear in our model at a more general stage than that at which we make contact with the special case of quantum mechanics. We discuss below how some recent work of Finkelstein's (1969, 1977, 1979) might allow such contact to be made.

The historical origins of the quantum theory concerned the experimental discovery of discreteness and an attempt to explain it using a continuum conceptual framework (we may consider that the Planck radiation formula was a striking experimental ratification of a theoretically arbitrary mathematical imposition of discreteness). Early quantum theory hardly claimed to be explanatory; the modern form of the theory has usually been seen as a successful reconciliation of the continuous and the discrete, and therefore as a satisfactory explanation of the latter. However, in view of the continuing unease with the conceptual foundations of the theory, it seems as appropriate today as it ever was to enquire (a) wherein the explanation lay, and (b) how successful it was. It is sensible to carry on our enquiry in the context of any of the traditional *Gedankenexperimente* (two-slit experiments, photon-splitting experiments, photon correlation experiments such as have been imagined by a sequence of theorists going back to Einstein, Podolsky, and Rosen).

As everybody knows, quantum theory has maintained that there is a distinct class of things in the universe called measurements or observations and that different rules apply to these from those that apply to interactions in which the acquisition of knowledge is not involved. In one way or another use is made of this principle to justify the importation into the formalism of a discrete principle. As everybody also knows, this principle has never produced peace of mind, even though the great thinkers of the quantum theory have concentrated their attention upon it. Consider, for example, the recent essay by Wheeler and Patton (1977). We shall refer to these arguments as the "observation metaphysic."

In Bohr's attempt to achieve an understanding of the observation metaphysic, an absolutely central part was played by his (Bohr's) insistence that all theoretical formulations had to be interpreted through the massively consistent and pervasive language which was at once classical physics and the common sense world. Bohr though it inconceivable that any underpinning or revision of this language using conceptual entities less evident to the senses was conceivable, practicable, or desirable. Indeed, his philosophy made a virtue of the necessity of this position.

In the *Ur* theory this position of Bohr's has been abandoned, though it would not be true to say that the "observer metaphysic" has gone with it. What has happened is that as a result of their finitist presupposition the *Ur* theorists have been able to present the conventional quantum theoretical view of measurement as a merely technical development free from its paradoxical characteristics, at the expense of a profound innovation in the application of probability to the quantum picture. The actual alternatives at any quantum process are finite, and the continuum of states out of which the measurement process picks one are in a different category, being "possibilities."

We, too, postulate entities that would be disallowed by Bohr's form of operationalism. We are equally concerned to find a comprehensible and still profound replacement for the "observation metaphysic," and claim to find it in the *individual* process.

Let us imagine a universe containing elementary entities which we may think of as our counterparts of the *Ur*s. To avoid confusion we will amend the terminology and call them *Schnurs* (German for "string")—a term that appropriately suggests computing concepts, in a way that represents their most fundamental aspect of concatenating strings. The Schnurs are discrete, and any representation they may have is two-valued. Consider a definite small number of them. Consider an elementary creation act as a result of which two different Schnurs generate a new Schnur, which is again different. We speak of this process as "discrimination." By this process, and by concatenations of this process, alone can the complexity of

the universe be explored. It is also necessary that a record of these discriminations and resulting creations be kept as a part of the structure defined by the Schnurs; otherwise there is no sense in saying that they have, or have not, been carried out. Hence we consider a new lot of Schnurs, which consist of concatenations of creation processes preserving the discriminate structure explored by the original Schnurs. The members of the new class are themselves constituents of the universe and are also free to take part in the creation or discrimination process, and to map up to higher or down to lower levels. This last requirement is the stage at which the necessity becomes clear for a reflexive or recursive aspect to our model, which in current quantum theory takes the form of the "observation metaphysics." The construction of a hierarchy of new levels of Schnurs is necessary to obtain an approximation to a physical continuum; by means of it we can ultimately speak of a physical entity in a background of other physical entities in accordance with the requirements of common sense. However, it makes no sense to speak of the individual entities except in terms of the part they play in the construction. Everything plays a dual role, as a constituent in a developing process, where something comes in from outside to interact, and as a synopsis or concatenation of such a process where the external interaction becomes subsumed in one new entity.

How can a thing be both aspects at once? We do not think we are able at present to say clearly how it can, and we must let our model, which incorporates this duality, lead us forward without having a complete insight, as earlier theorists had to do in quantum theory. However, we are in a better position than current quantum theory, for we can adopt a strictly process view and insist that we always view the process from one viewpoint—albeit a viewpoint that can, and must, change. Then we are freed from conceptual confusion, and we progress by considering stability conditions under which the limitations of our way of approaching the inescapable duality are compensated. Indeed, we find in the stability of the hierarchy levels a profound condition under which we can be sure of a sort of automatic self-consistency which reflects itself in the properties of quantum objects, and which is the basis of our interpretation of our model.

We do not think it impossible that a mathematical way of thinking will emerge in which the dual function can be comprehended without the device of considering the structure of the universe from one point at which the decision making is occurring. One might revert to a more classical or synoptic mathematics. However, we do not think we can do it at present [though Parker-Rhodes (1978), whose work has played such an important part in our model, and who feels uncomfortable with a process philosophy, is trying to formulate something very similar in terms of a "mathematics of

indistinguishables," which transcends the process aspect]. We would conjecture that if such a conceptual framework ever is discovered, its proper field of application would be wider than physics, and that the restricted process view would probably be adequate for physics.

Our view of space–time is *constructive* in the sense that there is one set of principles that gets us from the Schnurs to whatever approximation to the continuum of space we decide we need. Our scheme is also constructive in the sense that we require that any mathematical constructions that are needed to specify the attributes of any physical things, including the space continuum, shall also be so derived. In this sense the *Ur* theory is not constructive, and we have found our vital objection to it in this lack of constructivity. This use of the term "constructive" is stringent. We are, however, using it as in its *locus classicus*, Brouwer's theory of mathematical intuition (which also stimulated the development of intuitionist logic).

Brouwer's basic concept is that of the free choice sequence. The formal need for the free choice sequence is to construct the continuum adequately. For Brouwer, the constructions of mathematics have no absolute quality but are creations of the intellect, whose validity is relative to the state of mathematical understanding at a given epoch. They play a part in guiding the development of the free choice sequences. So do other considerations that we should normally regard as contingent. (An example of Brouwer's was to make the development of a free choice sequence depend upon whether, at the particular time in question, four successive sevens were known to occur in the expansion of π). It would be possible (and Brouwer was quite open to this suggestion) to regard the totality of considerations that could influence free choice sequences as including the contingent behavior of physical systems, in which case the similarity of the processes in our constructive model and the basic entities with which Brouwer constructed his universe would be quite close.

It would be fascinating to pursue this connection with Brouwer's thought, but this cannot be the place. We introduce it at all here only because it may be felt by some readers that our theory requires a mathematical ontology which is just wrong ; it may reassure the readers to know that something very like what we propose has been authoritatively put forward for analogous reasons in the literature of the foundations of mathematics. The connection is also relevant to our present discussion, because Brouwer's constructivism has no separate world of mathematical entities; the difficulty we encounter with the *Ur* theorists is that they allow themselves the use of continuous mathematical constructions where we felt that a constructive development should include mathematical entities used in the theory.

When one has a model for elementary processes one has to reconcile it with the macroscopic awareness of the world as an extended manifold of space and time. This is a large undertaking, which is usually not very explicitly faced. The traditional argument of a correspondence limit is only a small part of the problem for it presupposes that the problem has already been solved for the microscopic entities. Traditionally physicists rely on macroscopic experience to have universal application, and face the resulting confusions piecemeal. The position of the *Ur* theorists is not dissimilar for, as we have seen, they allow themselves to introduce continuum group theory, which then imports the principle of interpretation of extended spaces. We have left ourselves no such loophole, and the problem remains to be tackled.

Finkelstein's process approach also has to face this problem. Two arguments of his are relevant to it: (a) He has shown (Finkelstein, 1969) that the left–right moves of a dichotomous variable on a two-dimensional checkerboard generate, in the limit as the step size goes to zero, the full forward light cone of the Minkowski $(3+1)$-space. (b) Given any partial ordering relation, one can, by a theorem due to Galois, construct a lattice logic. If the lattice logic is that of bra and ket, then a theorem of Birkhoff's allows the construction of Hilbert space from the lattice (Finkelstein, 1979). It is not clear to us that this can meet the whole problem from our point of view; his approach might still end up with the commitment to macroscopic experience that we are trying to avoid. We hope that his treatment will turn out to be relevant to our problem. Certainly the lattice-theoretic result could be very significant in establishing a connection with Hilbert space.

We turn now to another difference between our Schnur theory and the *Ur* theory. This concerns the question whether we locate the reflexive character in the individual *Ur* processes or in statistical assemblages of them. We hold the former view, the *Ur* theorists the latter. The tradition is on our side, even though one is stretching a point in arguing as we have done that traditional quantum theory fails crucially at the point where it has to appeal to an observation metaphysic to introduce the reflexive character of quantum processes and yet claim support from that quarter. Still, the traditional argument that the essential character of quantum processes has to be defined for individual processes is very strong. One is accustomed to having to refute various facile approaches to the foundations of quantum theory by pointing out that the characteristic quantum-observation effect is individual and therefore cannot depend upon a statistical effect. For example, in the photon-splitting experiment, the incident beam can be attenuated to such a degree that the incident photons

would have to be treated individually, and therefore could not interfere. Yet interference does take place. This piece of experimental evidence provides a very sharp refutation of any view whose attribution of simple atomic properties to the photons is subject to the restriction that one may consider only statistical distributions of these; von Weizsäcker's distinction between possibility and probability in conjunction with his principle of the finite alternatives allowed by the *Ur*'s is used to explain why the *Ur* theory is not in this class [J. H. M. Whiteman (1971) introduced a concept that he called *potentiality* to achieve a similar end.] However, this matter is crucial and one feels that the detailed mechanics that makes a statistical effect appear as an individual one should be presented. We think that our model, in which the effect is individual, has a crucial advantage, and that this advantage is a direct consequence of our constructive approach.

In all other respects, we find ourselves in complete agreement with the analysis of the use of probability that von Weizsäcker (1978) has undertaken. Probability is closely related to the concept of time in the quantum physics context. The concept of time that is commonplace in modern philosophical writing, and which owes more to Hume than to any other thinker, seems to be in conflict with a good deal of the thinking of physicists. Starting with Galileo, the time of physicists is based primarily upon the analogy between time "displacement" and displacement in space. Our model has developed partly from discussion that was designed to show that in a discrete approach one might have the advantage of adopting the Humean point of view without outrage to physical theory. Then one could take the past simply as the fixed domain and the future as the domain of uncertainty and of probabilistic inference. This point of view can be tagged "Fixed Past, Uncertain Future" (Noyes, 1975, 1976, 1977).

It is obviously tempting to identify the duality of function of our elementary discriminators or Schnurs with the duality of description in complementarity. Certainly the two are connected, but the connection is not simple, as must be clear from the foregoing discussion of the differences between our view and current quantum theory. Bohr's view of complementary descriptions seems to be very much a special form of a more general philosophy and to have had its special form dictated by the special form in which quantum physics has developed. It is probably safe to say that if one could state the general philosophy without such special reference, it would contain the reflexive or recursive character with which our discussion has been concerned. However, Bohr's philosophy has proved notoriously difficult to state in this bare form in spite of the best efforts of fifty years. We conclude this section by stating what we feel to be the reason for this recalcitrance.

In a discrete or finite theory it is not too perplexing to introduce a reflexive philosophy by using a recursive mathematical model, which is what we do. The really perplexing difficulties seem to appear if we associate this reflexive character with an observation imagined against an objectively existing background, as is done in so-called "measurement theory." Two incompatible principles are being appealed to. One principle requires entities in the universe to be constructed using the observation process; the other takes a realist view of them. Not surprisingly, no reconciliation of the resulting perplexities is achieved by studies at a technical level where fundamental principles tend to be assumed rather than discussed.

One question has been avoided till now. In our model the elementary entities have a dual function. One of the dual aspects is analogous to that of an observing system. Do we imagine that this aspect of its dual role would correspond to the quantum theoretical "observation," and if so how would we react to those writers on quantum theory who wish to see something irreducably mentalist in the observation? In reply, we would first observe that we are not compelled to answer this question before we can use our model. We have a model for interactions which are elementary (*Ur*) in the sense that all we know is built up from them, and we have an interpretation for the model in terms of scattering processes. This interpretation does not have to be the only one. We have tacitly assumed that the conditions of high energy are favorable for exhibiting the simplicity of the model and hence the scattering situation. However, under other conditions the interacting entities might even be living organisms with consciousness. The model should still apply. What we absolutely are not either compelled or allowed to say is that the phenomenon of consciousness as a separable ingredient is necessary for the interaction.

2. CONSTRUCTION OF THE HIERARCHY

In this section we develop the specific formalism by which we are implementing the program discussed above, using a very explicit representation of the abstract hierarchical structure. The mathematical structure itself is developed in group theoretic language in the Appendix. Our basic elements are the existence symbols 0 and 1, and our basic mathematical operation is symmetric difference or addition modulo2: ($0+0=0$, $1+0=1$, $0+1=1$, $1+1=0$). The symbols are grouped as ordered sets ("columns") of height n ($n=1,2,3,\ldots$). The comparison between two such columns is called "discrimination." Each column x, whose height we can

indicate by writing $(x)_n$ for x, has elements ("discriminators") x_i ($i = 1,\ldots,n$); thus $x=(x_i)_n$. A column with every element $x_i=0$ is called a "null" column. The basic binary operation of discrimination between two columns x,y of equal height is defined by

$$D_n(x,y)=x+y=(x_i+y_i)_n \tag{2.1}$$

The concept of such discriminators is abstracted from the more familiar idea of discrete quantum numbers, while the discrimination operation itself can be viewed, as we will discuss in another section, as an abstract model of a general scattering ("production") process in which the result of scattering two different systems is a third system that differs from either. Our mathematical model thus describes chains of atomic or elementary processes. Our policy for presenting the theory is first to establish a correspondence between the mathematical model that describes these chains of processes and the familiar structure of quantum numbers. In this way we can first view the mathematical model as providing a classification scheme. The basic dynamics of our theory is represented during the construction of this classification scheme by the concept of *discriminate closure*. We introduce this concept by the following argument.

Starting with columns of a given height, we imagine new columns formed by concatenating a sequence of them. Entities corresponding to the new columns are said to constitute a new level in the hierarchy. There is no difference between the new and the old in logical type; the only difference is that the boundary between the observing system and that which is observed has changed. The great conceptual and mathematical difficulties of such an idea can be handled in one special case, which is therefore of great importance. This case is that in which the entities at the new level represent all combinatorially possible concatenations of entities at the previous level, starting with a given set. Hence we get a *discriminately closed subset*.

A "discriminately closed subset" or DCsS consists of one or more nonnull columns, such that discrimination between any two distinct columns in the set yields a member of the set. Assume that we start from a basis of j linearly independent columns, that is, columns for which no sum of two or more different columns is null. Then there will be 2^j-1 distinct discriminately closed subsets. Symbolizing a DCsS by { }, a basis of two columns a,b gives the three DCsSs $\{a\}$, $\{b\}$, $\{a,b,a+b\}$; a basis of three columns a,b,c gives the seven DCsSs $\{a\}$, $\{b\}$, $\{c\}$, $\{a,b,a+b\}$, $\{b,c,b+c\}$, $\{c,a,c+a\}$, $\{a,b,c,a+b,b+c,c+a,a+b+c\}$. Proof of the general result is immediate either by noting that the number of DCsSs is simply the number of ways we can combine j things $1,2,\ldots,j$ at a time, or by

induction. The first step in constructing the hierarchy is then to consider the $2^j - 1$ DCsSs so formed as the basic entities of a new level.

The reason for seeking a constructive process of hierarchical nature that yields levels of rapidly increasing (in our case exponentiating) complexity is again abstracted from experience. We have detailed in the first section the reasons why we start from an elementary process (discrimination) which already implicitly contains the "observation metaphysic." There we also explained why, in our view, we adopt a constructive, process-oriented approach. The further requirement that the hierarchy so generated terminate is a basic requirement if we are to retain the principle of finitism. We defer the discussion of the reflexive character of the scheme until it is further developed. That the *combinatorial hierarchy* obtained by starting with columns of height $n=2$ yields levels of interesting physical structure and sufficient complexity, and terminates at the appropriate level, has been shown previously (Bastin, 1966). We summarize the construction here.

We have seen that, given j linearly independent columns, we can always construct $2^j - 1$ DCsSs at that level. For them to form the basis of a new level, however, they must themselves be representable by linearly independent entities that contain the same information about discriminate closure as the sets themselves. For this purpose we introduce multiplication modulo 2 and matrices because linear operators preserve discrimination. We look for $2^j - 1$ matrices which (a) map each column in one of the subsets onto itself and onto no other column; (b) map only the null column onto the null column, and hence are nonsingular; and (c) are linearly independent. Provided this can be done, and the original basis consists of columns of height n, then the matrices themselves can be rearranged as columns (e.g., by putting one row on top of another by some consistent rule), and will then provide a linearly independent basis of $2^j - 1$ columns of height n^2. Such mapping matrices are easy to find for $n=2$ (see below). Explicit examples have been found for $n=3$, 4, and 16 (Noyes, 1978) proving the existence of the hierarchy. A formal existence proof has also been provided (Kilmister, 1978) based on unpublished work (Amson, 1976).

The use of matrix algebra could be misunderstood as implicitly incorporating into the scheme the basic assumptions of linear algebra. In fact, matrix algebra using the symbols 0, 1, discrimination, and multiplication mod 2 is the natural extension of the discrimination idea to incorporate mappings. This can be seen in more formal terms by following the group theoretic discussion given in the Appendix.

We can now present the general situation. We have seen that if at some level l there are $j(l)$ linearly independent columns of height $n(l)$, we

can construct immediately $d(l)=2^{j(l)}-1$ DCsSs. Provided these can be mapped according to the restrictions given above, they form the basis for a new level with $j(l+1)=d(l)$ and $n(l+1)=n^2(l)$. The process will terminate if $n^2(l)<2^{j(l)}-1$ since at level l there are only $n^2(l)$ linearly independent matrices available; clearly this will always happen for some finite n. The situation for $n(1)=j(1)=N$, i.e., when the vectors at the lowest level which span the space are used as the basis, is exhibited in Table I. Thus, perhaps surprisingly considering the simplicity of the assumptions, the hierarchy with more than two levels turns out to be unique. See the Appendix for a more detailed discussion.

Although the cardinal numbers given by the hierarchy are unique, the specific representations used in the construction are not. It is important to understand this clearly because it is a complication in making any *simple* interpretation of the discriminators as representing the presence or absence of particular conventional quantum numbers in an isolated system. This ambiguity is present at the lowest level since for the two basis columns we have three choices: $a=\binom{1}{0}$, $b=\binom{0}{1}$; $a'=\binom{1}{0}$, $b'=\binom{1}{1}$; $a''=\binom{1}{1}$, $b''=\binom{0}{1}$. Corresponding to these three possible choices of basis, there are three different sets of mapping matrices. When, as here, the number of independent columns is equal to the height of the columns ($n=j$), the maximal discriminately closed set (MDCS) contains all the nonnull vectors in the space [here it is $\{\binom{1}{0},\binom{0}{1},\binom{1}{1}\}$] independent of the choice of basis; further, the only possible mapping matrix for the MDCS is then the unit matrix. For the first basis, the mapping matrices for $\{a\}$ and $\{b\}$ are $\binom{11}{01}$ and

TABLE I. The Possible Hierarchies Starting from $n(1)=j(1)=N$

	l	1	2	3	4	Hierarchy terminates because
$N=2$	$n(l)$	2	4	16	256	
	$j(l)$	2	3	7	127	$(256)^2<2^{127}-1$
$d(l)=2^{j(l)}-1$		3	7	127	$2^{127}-1\approx 10^{38}$	
$N=3$	$n(l)$	3	9			
	$j(l)$	3	7			$9^2<127$
	$d(l)$	7	127			
$N=4$	$n(l)$	4	16			
	$j(l)$	4	15			$16^2<2^{15}-1$
	$d(l)$	15	$2^{15}-1$			
$N>4$						$n^2(1)<2^{j(1)}-1$

$\binom{10}{11}$, respectively. For the second $a = a'$, so that matrix is the same but the mapping matrix for $\{b'\}$ is $\binom{01}{10}$; for the third we note that $a'' = b'$ and $b'' = b$. Rearranging the matrices as columns then give three different possible bases for the second level of the hierarchy, namely, with the rule

$$\binom{AC}{DB} \rightarrow \begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix}$$

$$a_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad b_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \quad c_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}; \quad a_2' = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad b_2' = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \quad c_2' = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix};$$

$$a_2'' = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad b_2'' = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \quad c_2'' = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix} \tag{2.2}$$

In addition to this ambiguity, there is the further problem that we could have used any other rule for converting the matrices into column vectors, provided only the same rule is used for all three matrices. Thus the ordering of the rows has no significance, and *within* a level the properties of the system under discrimination are unaltered by a permutation of rows in the basis. An important structural property which does emerge, however, is that instead of the basis of three unit columns such as (1000), (0100), (0010), or any linearly independent set constructable on such a basis, at least two of the columns in the basis always contain two ones in the same two rows. This property guarantees that the MDCS (up to a permutation of rows) at the second level will always be

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right\} \tag{2.3}$$

Note that the first two rows may always be written as $\binom{1}{1}$ or $\binom{0}{0}$. We shall find this fact significant as a clue to physical interpretation. (Note that "rows" always refers to places in a column even though columns may be printed vertically or horizontally for purely typographical reasons.)

When it comes to constructing mapping matrices for the second level, we cannot use the unit matrix to represent the MDCS given in equation

(2.3) because it maps all 15 possible nonnull columns of height 4 onto themselves, and not just the required seven. The eight columns that must be excluded are of the form $(10xy)$ or $(01xy)$. A nonsingular matrix that has none of these as eigenvectors, but all the columns of equation (2.3), is exhibited in equation (2.4):

$$a\begin{bmatrix}0&1&0&0\\1&0&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}\begin{bmatrix}x\\x\\y\\z\end{bmatrix}=\begin{bmatrix}x\\x\\y\\z\end{bmatrix}, \qquad \begin{bmatrix}a\\b,c,d\\e,f,g\end{bmatrix}\begin{bmatrix}x\\\bar{x}\\y\\z\end{bmatrix}\neq\begin{bmatrix}x\\\bar{x}\\y\\z\end{bmatrix},$$

$$\bar{x}=1+x, \qquad \begin{vmatrix}a\\b,c,d\\e,f,g\end{vmatrix}\neq 0$$

$$b\begin{bmatrix}0&1&0&0\\1&0&0&1\\0&0&1&1\\0&0&0&1\end{bmatrix}\left\{\begin{bmatrix}1\\1\\0\\0\end{bmatrix}\begin{bmatrix}0\\0\\1\\0\end{bmatrix}\begin{bmatrix}1\\1\\1\\0\end{bmatrix}\right\}c\begin{bmatrix}0&1&0&0\\1&0&1&0\\0&0&1&0\\0&0&1&1\end{bmatrix}\left\{\begin{bmatrix}1\\1\\0\\0\end{bmatrix}\begin{bmatrix}0\\0\\0\\1\end{bmatrix}\begin{bmatrix}1\\1\\0\\1\end{bmatrix}\right\}$$

$$d\begin{bmatrix}1&1&0&0\\1&0&0&0\\0&0&1&0\\1&0&0&1\end{bmatrix}\left\{\begin{bmatrix}0\\0\\1\\0\end{bmatrix}\begin{bmatrix}0\\0\\0\\1\end{bmatrix}\begin{bmatrix}0\\0\\1\\1\end{bmatrix}\right\}e\begin{bmatrix}0&1&1&1\\1&0&0&1\\0&0&1&1\\0&0&0&1\end{bmatrix}\left\{\begin{bmatrix}1\\1\\0\\0\end{bmatrix}\right\}$$

$$f\begin{bmatrix}0&1&0&0\\1&1&0&1\\0&0&1&1\\0&1&0&1\end{bmatrix}\left\{\begin{bmatrix}0\\0\\1\\0\end{bmatrix}\right\}g\begin{bmatrix}0&1&0&0\\1&1&1&0\\1&1&1&0\\0&1&1&1\end{bmatrix}\left\{\begin{bmatrix}0\\0\\0\\1\end{bmatrix}\right\} \tag{2.4}$$

Choosing as a basis the columns (1100), (0010), (0001) we also exhibit six specific mapping matrices which have as eigenvectors only the columns in the six remaining DCsSs. This representation is not unique, since we find that of the 35 possible choices of three columns as a basis, omitting those that are not linearly independent or that are equivalent to others under a permutation of rows, there are 15 alternative choices. However, all of them have more than four descriptors in the three columns, so the choice exhibited is in that sense the simplest.

In order for these seven mapping matrices to form a basis for constructing the $2^7-1=127$ DCsSs of level III, they must be linearly independent. The linear independence is exhibited explicitly in equation

(2.5), after using a particular rule for rearranging the matrices as columns:

$$\begin{array}{c} \\ \begin{bmatrix} I & A & K & L \\ B & M & G & E \\ O & P & C & F \\ J & N & H & D \end{bmatrix} \end{array} \rightarrow \begin{array}{c} \\ \begin{bmatrix} A \\ B \\ C \\ D \\ E \\ F \\ G \\ H \\ I \\ J \\ K \\ L \\ M \\ N \\ O \\ P \end{bmatrix} \end{array} \begin{array}{c} a \\ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \end{array} \begin{array}{c} a+b \\ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \end{array} \begin{array}{c} a+c \\ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \end{array} \begin{array}{c} a+d \\ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \end{array} \begin{array}{c} b+e \\ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \end{array} \begin{array}{c} b+f \\ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \end{array} \begin{array}{c} b+c+f+g \\ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \end{array} \tag{2.5}$$

The explicit choice of mapping matrices, which again is not unique, was again made in such a way as to get the simplest possible basis for level III in which all 16 rows are occupied.

To construct level IV we first find a basic matrix that has any of the 127 vectors that can be constructed from the seven given in equation (2.5), and none of the remaining 128 nonnull columns of height 16 that are not of this form, as eigenvectors. One possibility is exhibited in equation (2.6):

$$\begin{bmatrix} 0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&1&1&0&0&1&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0 \end{bmatrix} \begin{bmatrix} a \\ a \\ a \\ a \\ b \\ b \\ c \\ c \\ d \\ d \\ e \\ e \\ f \\ f \\ g \\ g \end{bmatrix} = \begin{bmatrix} a \\ a \\ a \\ a \\ b \\ b \\ c \\ c \\ d \\ d \\ e \\ e \\ f \\ f \\ g \\ g \end{bmatrix} \tag{2.6}$$

One then forms the 127 DCsSs, and finds nonsingular mapping matrices for each of them. This is done by leaving the first six rows of this basic matrix untouched—which guarantees that none of the unwanted columns from the 128 are brought back as eigenvectors—and adding ones one at a time to the remaining structure in such a way as to restrict the eigenvector set. Care must be used not to make the matrix singular and to maintain linear independence. The procedure is straightforward, if somewhat tedious, so the explicit result will not be given here. This empirical procedure thus proves the existence of all four levels of the hierarchy.

3. LEVELS 0, I, II, AND III: BARYONS, MESONS, LEPTONS, AND PHOTONS

In this section we attempt to correlate the mathematical structure developed above with some facts known from elementary particle physics. Because any physical process requires development of the hierarchy through the levels successively, the significant physical magnitude is not the cardinal of each level separately, but rather their cumulative sum, which gives the sequence $3, 10, 137, 137+2^{127}-1 \approx 10^{38}$. Obviously these numbers could be interpreted immediately as the inverse of the superstrong, strong, electromagnetic, and gravitational coupling constants and suggest that in some sense the cumulative levels refer to systems of bosons with increasingly refined definitions of their possible interactions. One way to make this more specific would be to assume that the various systems at each cumulative level all have equal *a priori* probability, and that the probability of "coupling into" any one of them by the characteristic described at that level is therefore the inverse of the corresponding number. We will give this vague idea of coupling more specific content shortly. Further, the fact that the first three levels can be mapped up or down freely, but that any attempt to construct a linearly independent representation of the fourth level with $2^{127}-1$ DCsSs must fail after $(256)^2$ linearly independent matrices have been selected, suggests that the destabilization of particle systems due to weak decay processes with coupling constant $10^{-5}m_p$ might also emerge from the scheme since $1/(256)^2$ has approximately this value (Bastin, 1966). This requires us to assume that the unit of mass in the scheme is the proton mass, but this is already clear from the initial sequence, since $\sim 10^{-38}$ *is* the gravitational coupling between two protons; the gravitational coupling constant between two electrons is 10^{-44}. Thus we can hope to derive the ratio of the electron mass to the proton mass once the scheme is sufficiently developed.

We are now in a position to state our policy toward the general question of the physical interpretation of the hierarchy so as to be

consistent with the identification that has already been made of the basic scheme of cardinal numbers with dimensionless constants. This policy has two aspects. First we have the task of identifying the quantum numbers with configurations in the hierarchy, and secondly, we have to introduce fields corresponding to the quantum numbers, and show that we should expect these fields to have the characteristics that we find in nature. The second task will be coterminous with that of defining an extended space for the particles to be "in." Only the first task is confronted in this section.

We wish to identify places in columns with quantum numbers, and we wish to regard associations of quantum numbers in given columns as *systems* which, under conditions of stability that have yet to be established, will carry over unchanged into stable or unstable particles. For the moment we call them "systems." These ends require us to solve the following problems:

(1) How to get an initial distinguishing characteristic of a column which is available for taking the first step in interpretation in the sense that it cannot be eliminated by choosing a different basis.

(2) How to interpret the interrelations of columns in a set (including of course a DCS) at one level.

(3) How to relate the interpretation of a column at one level with that of columns of different lengths and hence different quantum numbers at another level.

The first step in the solution of these problems is to define *conservation* in respect of a set of properties to each of which a quantum number is conventionally assigned. These are the eight properties (1) of having z component of spin up, (2) of having z component of spin down, (3) of having charge $+$, (4) of having charge $-$, (5) of being a lepton, (6) of being an antilepton, (7) of being a baryon, (8) of being an antibaryon.

It makes things clearer to begin by speaking of properties and only later of the dichotomous variables that can correspond to quantum numbers. The latter require two rows to represent them.

Definition. A quantum number will be said to be *conserved* if the algebraic difference between the number of ones in the corresponding pair of rows of properties is constant *at each step* in the generation process.

The choice of the foregoing definitions (in particular that of the conserved quantities and of their relation to descriptors) embodies a lot of detailed argument whose correctness must be judged by the coherence of the resulting scheme. Moreover the choice of quantum numbers assumes the emergence of discrete quantities through the history of the quantum theory so that the theory is now at a stage that makes it ripe for combinatorialization. Thus in particular the use of the z component of spin as the appropriate quantum number for combinatorialization is obscured

by the spatial idea of spin, but it is becoming more perspicuous as the "helicity state"; we stick to the earlier term.

Our definition of conservation introduces two novelties in principle. One is that of forming the algebraic sum of a set; the other is that of a primitive notion of simultaneity. The two are related since in forming the sum one is making an assertion about what is true collectively of the set *at each step*. The change is considerable since one abandons the principle of individual access in enumeration; although this change is already implicit in the hierarchy construction itself, it is appropriate to introduce it here with the motivation of the very fundamental idea of conservation. It is also at this point that we see the root of an idea of sequential delay that will take us from a purely sequential theory to one with a more conventional space and time. However, having recorded this starting point, we will not attempt to develop it further now.

We notice in the above account that all the structures that are going to be interpretable (dichotomous variables) require representation by *two* rows, so that systems of one row are not given a meaning. This principle already exists as a matter of logical necessity in the hierarchy construction —a correspondence that indicates satisfactory coherence in the theory as a whole. (The level of single elements cannot generate a hierarchy.) It follows that at level II the 4-columns are properly regarded as a pair of pairs.

We now return to the set of problems posed above. To handle the first problem—that of initial interpretation—we first draw attention to the "doubled discriminators" of Section 2, which we have shown must exist in the mapping construction. We note first that this asymmetry is already enough to refute the criticism that since one can always take a minimal basis using only columns of the form

$$\begin{array}{ccccc} 0 & 0 & & & 1 \\ & & & & 0 \\ \vdots & \vdots & \cdots & & \vdots \\ & 1 & & & \\ 1 & 0 & & & 0 \end{array}$$

and since row position is arbitrary, no interpretation that depends on relative position of ones can be significant. In fact we cannot always take such a minimal basis; we are therefore justified in beginning our interpretation with a nonminimal basis, and in particular with one in which we have the doubled discriminators which we have shown to be necessary.

The doubled discriminators enable us to develop a notation for putting together two systems, each of which is described by a dichotomous variable; we shall use spin as our first example (Bastin, 1976a). In Table II

TABLE II. Triplet–Singlet System from Two Dichotomic Vectors

Conventional notation				Hierarchy notation			
$S=1$			$S=0$	$S=1$			$S=0$
$S_z=1$	0	-1	$S_z=0$	$S_z=1$	0	-1	$S_z=0$
$\begin{bmatrix}1\\0\\1\\0\end{bmatrix}$	$\frac{1}{2^{1/2}}\begin{bmatrix}1\\1\\1\\1\end{bmatrix}$	$\begin{bmatrix}0\\1\\0\\1\end{bmatrix}$	$\frac{1}{2^{1/2}}\begin{bmatrix}1\\-1\\-1\\1\end{bmatrix}$	$\begin{bmatrix}1\\0\\1\\0\end{bmatrix}$	$\begin{bmatrix}1\\1\\1\\1\end{bmatrix}$	$\begin{bmatrix}0\\1\\0\\1\end{bmatrix}$	$\begin{bmatrix}0\\0\\0\\0\end{bmatrix}$

we assign the first two rows to indicate spin up or spin down of one system, and the second two rows to refer to spin up or spin down for the other system. The resulting singlet or triplet states are represented in Table II both in the conventional notation using algebraic vectors and in the hierarchy notation using only existence symbols. We see that the descriptive content of the two notations is identical so far as distinguishing the four possible singlet or triplet states goes. We also note that the singlet state is the null column; we can only give meaning to such a state in a richer system with more rows containing nonnull descriptors.

The spin-z state refers to a *single* system of spin $1/2$ (which can be up or down) for which the algebraic notation is $\binom{1}{0}$ or $\binom{0}{1}$. A singlet/triplet system is either a spin-0 system with one state and a spin-1 system with three states, or the composition of two spin-$1/2$ to give the same result. The conventional notation for the result is given in Table II in comparison with our notation.

We have shown the above identification to be possible and consistent with the idea of conservation; we have not shown it to follow necessarily from the existence of doubled descriptors. The latter demonstration requires a new physical principle. In addition to the association of two existence symbols to make one dicotomous variable, we have encountered the association of two identical existence symbols in two rows to make an effective single existence symbol—a development that was forced by the necessary occurrence of "doubled descriptors." If we were to exploit, at level II, the full possibilities of the increased scope in our descriptive language offered by treating each row as independent, we would get $16-1=15$ possible systems. The mapping construction allows only 7 of these, consisting of one doubled existence symbol and one dichotomic variable. We see this more clearly if we enumerate four cases:

(I) Triple existence symbol plus single existence symbol. This is isomorphic to the basis for level I and gives nothing new.

(II) Triplet system exhibited in Table II. Since we are considering a system of four rows, the singlet possibility effectively represents nothing

and is excluded. Further, we see that triplet system is simply a doubled representation of level I and again gives us nothing new.

(III) The doubled existence symbols together with the two-row dichotomous variables already exhibited in (2.3) is the unique MDCS forced by the hierarchy construction.

(IV) The maximum set, obtained by treating each descriptor independently, is excluded by the hierarchy construction. We are thus limited to case III, which exhibits the necessity of the interpretation.

We have used conservation as the basic interpretive principle. We have yet to display this in the context of the sequential dynamics of step-by-step discrimination which is implied in speaking of conservation. We approach this problem for the first three levels of the hierarchy by interpreting sequences of discriminations as the flow of quantum numbers through sequentially ordered "Feynman diagrams." As we will see below, the direction of the sequence has to be established *external* to the hierarchy as part of our construction of a finite representation of "space–time" which could, sometimes, approach conventional space–time in a large number regime.

The basic postulate by which we convert the symmetric discrimination operation $x+y=z=y+x$ into a partial ordering is to assume that when the discrimination occurs between two identical columns, i.e., when $x+x=0$, there is some externally established criterion, which eventually is to be established recursively, by which the two x's are assigned to *different* sets. Our justification for this assumption is our equally basic postulate that the nonnull descriptors in a column refer to *conserved* quantum numbers. This is clearly impossible in the case at hand if both columns are on an identical footing, since then the symmetric operation would destroy quantum numbers. Abstracting from the empirical structure of elementary particle physics, we assume that $x+x=0$ refers to a particle and an antiparticle which, so far as quantum numbers go, can indeed annihilate each other if they have opposite charge, opposite baryon number, opposite lepton number, and equal but opposite helicity ("z component of spin"), and *no* other distinguishing characteristics. This idea has yet to be worked out in deductive mathematical terms. Here we work it out, level by level through the first three levels of the hierarchy, using a diagrammatic technique abstracted from the familiar rules for Feynman diagrams.

Consider first a "universe" consisting only of identical columns x. This we call "level 0" of the hierarchy, since it clearly can be modeled by sets of "columns of height one" consisting of sets of the existence symbol 1, or the null 0. Notationally we represent the basic discrimination $1+1=0$ by Figure 1a, where the first 1 stands for a "particle" represented by the solid line and the second 1 stands for an "antiparticle" represented by the

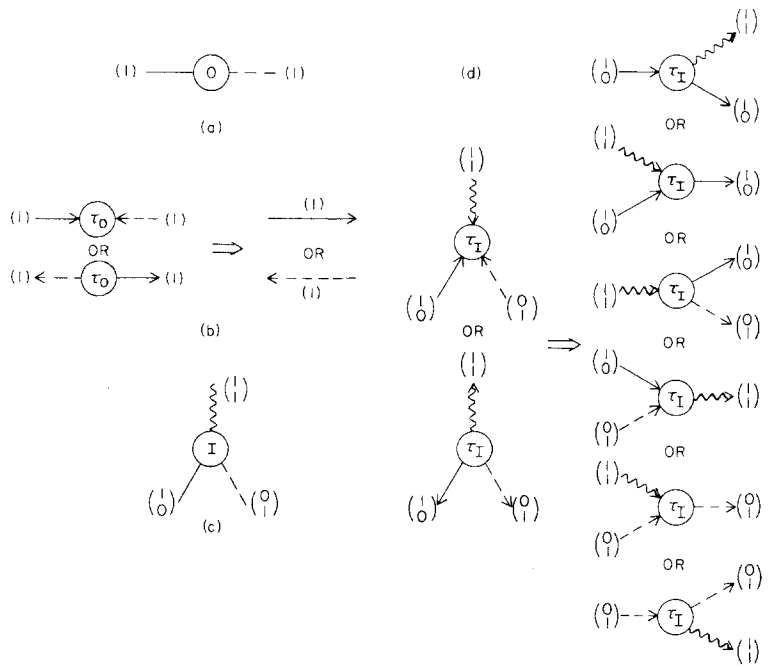

Fig. 1. (a) Level-0 discrimination interpreted as a particle–antiparticle diagram. (b) Level-0 sequentially ordered discrimination as particle or antiparticle. (c) Unique level-I DCsS as a particle–antiparticle–quantum vertex. (d) Sequential ordering of the level-I vertex.

dashed line. To make this into an ordered relation we assume each line has a "direction" relative to some externally established sequence of discriminations, which we indicate by placing an arrowhead on the line. The interpretation that conserves the (single type of) quantum number is then that both lines are "incoming" or "outgoing" as in the left side of Figure 1b. We now adopt the Feynman rule that a particle moving "forward" is the same as an antiparticle moving "backward." Since there is only one type in our level-0 "universe" we reverse one arrow. Then the discrimination has no effect and the universe consists only of particles moving forward (or of antiparticles moving backward). This is truly a Parmenidean universe in which there are no scatterings and nothing happens. In a broader context with higher columns, our (now ordered) discrimination makes a partial ordering selecting sets of "identical" columns with an "orthogonality" relation. So far as we can see, if this were all we had, we would have precisely the model discussed by Finkelstein (1977, 1979)—an "emission" followed by an "admission" which taken together form a

detection which, when null, corresponds to the failure to detect—a partial ordering in which the "particles" continue to "move" undisturbed. But our structure is more complicated, as we will see shortly. An alternative to introducing an ordering relation might be to develop a metalanguage in which we can retain the nonordered discrimination operation; as we understand it, this is what Parker-Rhodes (1978) has done in his theory of indistinguishables, which allows the discussion of "twins" that are individually indistinguishable, and that cannot be ordered, but which allows the assignment of cardinal numbers to sets composed of them. We believe the route followed here is more consistent with our basic process philosophy.

We now proceed to level I, where we have three nonnull columns that can be symbolized as follows: $\binom{1}{0}$ by ———, $\binom{0}{1}$ by ----, and $\binom{1}{1}$ by ∿∿∿. We can symbolize the unique MDCS by the discrimination diagram given in Figure 1c. As a nonordered discrimination diagram this is to be interpreted as representing the fact that discrimination between any two of the columns yields the third. As a Feynman diagram with all three lines either incoming or outgoing (Figure 1d) there is still no internal way to assign order. If, however, we assign a sequential direction externally, and use the Feynman rule, we obtain six possibilities also given in Figure 1d. There is now a structural difference compared to "level 0," because we now need a rule to say what happens to the two rows when we reverse the direction of the arrow. We see that to conserve quantum numbers we must interchange the two rows.

Now physical interpretation becomes possible. Row one represents one dichotomic variable, or conserved quantum number, whose presence or absence is indicated by the exitence symbols 1 or 0, respectively. The second row represents a second distinct dichotomic variable. In order that both quantum numbers be conserved, they must be conjugate in the sense that reversing sequence interchanges rows. The simplest choice for interpretation, following Eddington's insight that the basic quantization is that of charge, is that the two quantum numbers are simply positive and negative unit electric charge. Then our rule that reversal of sequence must be coupled to interchange of rows translates to the usual Feynman rule that a positive particle moving forward "in time" is equivalent to a negative particle (antiparticle) moving backward "in time." Note that in contrast to $\binom{1}{0}$ and $\binom{0}{1}$ the column $\binom{1}{1}$ is self-conjugate, and we are free to assign it either direction until we have sufficient *external* structural information to specify that direction in another way.

High-energy physics allows us to provide an experimental model isomorphic to this lowest level of the hierarchy—a hydrogen bubble chamber in a magnetic field with a beam of antiprotons incident. Protons curve one way and antiprotons the other way, distinguishing the two

quantum numbers. Annihilation produces electrically neutral quanta that leave no tracks in the chamber but whose presence can be inferred by the appearance of proton–antiproton pairs that can be spacially correlated with kinks in the tracks. A more detailed working out of this operational definition of quantized particles has been given elsewhere (Noyes, 1957). Such experiments provide direct empirical evidence for the quantization and conservation of unit electric charge. Note that the direction of the tracks must be inferred from external information on which side of the chamber the beam enters, or internally by the relation between density of bubbles along the track and velocity or energy. Relativistic kinematics allows the specific case when particle, antiparticle, and neutral quantum all have the same mass to be made into a model in which the three bind ("bootstrap") to form a single particle of the same mass and charge as one of the three, as has been discussed elsewhere (Noyes, 1979).

By such external considerations, we can talk about the ordered vertices of level I, symbolized in the figure by $\overline{\tau_{\mathrm{I}}}$ since they will eventually become time ordered scattering vertices, and the nonordered I, which represents the unique DCsS of level I. But it is easy to see that if we start with an arbitrary statistical assemblage of all three possible columns and all "directions," on the average nothing will happen. There will be the (unobservable and ignorable) discriminations of "level 0," and level I vertices with as many incoming as outgoing lines. Charge is conserved in the microscopic processes, and hence for the system as a whole. Any asymmetries would have to be established externally. This "universe" is still Parmenidean so far as observable consequences go.

When we go to level II the situation changes. If we represent columns $(11xy)$ by ⩻, and columns $(00xy)$ by ∿∿ the seven DCsSs of Equation (2.4) can be pictured by the seven discrimination diagrams given in Figure 2. To convert these to ordered diagrams that conserve quantum numbers, we see that rows 3 and 4 are isomorphic to level I, and that we must interchange these two rows when we reverse the direction of an arrow. But rows 1 and 2 are self-conjugate and act within this group of seven columns like a single new dichotomic variable which is either present or absent. But now we have eight additional columns $(10xy)$ and $(01xy)$ *outside* the hierarchy. Under our basic statistical assumption that initially all columns have equal probability, and that all rows are to be interpreted in terms of conserved quantum numbers, we see that we have added not one but two new dichotomic variables. Further, they also can form DCsSs, as we can see for example in Figure 3, ignoring for the moment the arrows. If, as we did within the hierarchy, we assigned all lines as incoming (or outgoing) the quantum number in the first row would be annihilated, contrary to our basic interpretive postulate. Hence, for these new vertices we *must*, in order

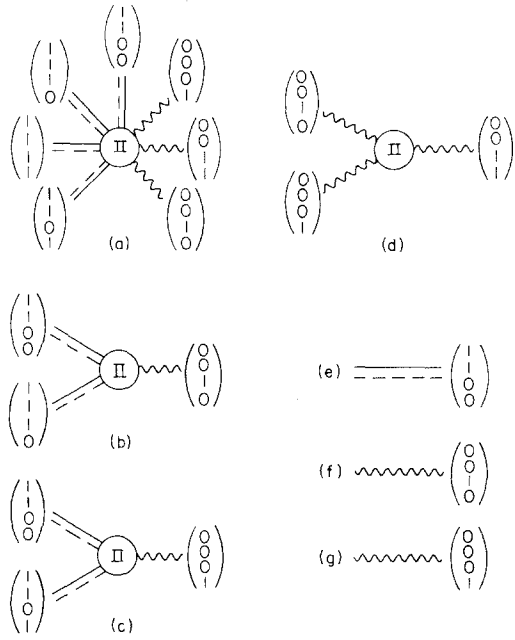

Fig. 2. (a) Unique level-II MDCS as a discrimination vertex within the hierarchy. (b)–(g) Six DCsS of level II in a particular representation as discrimination vertices within the hierarchy.

to conserve quantum numbers, assume that we have one incoming and two outgoing arrows. Then for any sequence of processes involving all 15 nonnull columns, *and* ordered vertices when (10*xy*) or (01*xy*) are involved, all four dichotomic quantum numbers will be conserved, provided (in our specific representation) we interchange both row 1 with row 2 *and* row 3 with row 4 when we change the direction of an arrow.

How are we to interpret this situation physically? We claim that the structural characteristic of the hierarchy—which, as proved in the last

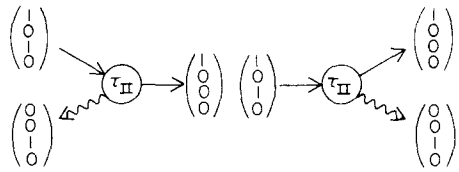

Fig. 3. Two level-II vertices outside the hierarchy sequentially ordered to conserve quantum numbers; the two shown can be interpreted as the emission or absorption of a charged quantum by a baryon.

section, necessarily doubles one descriptor in the basis used for constructing level II—implies two different types of dichotomic quantum numbers at level II. Since one pair is isomorphic to level I, we retain the identification of this with electric charge. The second we propose to interpret as baryon number in row 1 and antibaryon number in row 2. The seven columns within the hierarchy can then be interpreted as four baryon–antibaryon pairs $B^+\bar{B}^0, B^+B^-, B^0B^-, B^0\bar{B}^0$, and three quanta $Q^+, Q^\pm, Q^-$. We also have the start of a sequential dynamics because of the necessity of ordered vertices once we deal with single baryons. If we identify these with nucleons and antinucleons, and the quanta with charged and neutral mesons, we have a crude model for nuclear physics. We will not develop this here, as we have a more specific calculation closer to reality to present in level III. Note also that, again by use of the bubble chamber, it is possible to distinguish baryons from mesons since, empirically, the number of baryons minus the number of antibaryons is conserved, while the number of mesons is not; charge is, of course, still conserved. A little thought should convince the reader that our rules guarantee this contact with experiment.

We wish to emphasize here the structural features of the hierarchy which allow this comparison with experiment. Suppose we ignored the doubling of one of the descriptors in going from level I to level II. Then we could model the $2^3-1=7$ DCsSs with columns of height 3, e.g., the basis (100),(010),(001) which would put all three rows on the same footing. Quantum numbers could be conserved, and the DCsSs mapped by 3×3 matrices, which would then give $2^7-1=127$ DCsSs of columns of height 9. But these cannot be mapped by the $9\times9=81$ linearly independent matrices available; this hierarchy terminates too quickly. Another alternative would be to use the $2^4-1=15$ DCsSs constructed from columns of height 4, e.g., using the symmetric basis (1000),(0100),(0010),(0001), which again makes all rows indistinguishable. These again can be mapped and provide a basis for $2^{15}-1$ DCsSs of columns of height 16; again these cannot be mapped by the $16\times16=256$ linearly independent matrices available, so this also terminates too quickly. Only the asymmetric basis obtained from the mapping of level I allows the continuation to both level III and level IV. Further, as we have seen, this asymmetric basis, by distinguishing between columns inside and outside the hierarchy allows us, for the first time, to introduce meaningful sequence along with conservation. Thus, discrimination, conservation, and the existence of DCsSs that can be mapped at a single level are even conjointly not enough. We *must* use the unique hierarchy construction to get a rich enough physics without additional postulates. When we do, we are rewarded by finding a structure that can be interpreted as exhibiting the asymmetry between baryons and mesons that lies at the core of all nuclear theory.

Going now to level III, we have seen in equation (2.5) that we have a representation of the seven basis columns with one quadrupled and six doubled descriptors. Here we are not on so firm ground in interpretation, because this representation is no longer unique, and we have to argue instead that it is the simplest and most symmetric representation we can construct. But we believe, though we have not proved, that *any* level-III representation will have a quadrupled descriptor. Guided by the hypothesis that the third level, being stable, should contain quantum numbers corresponding to the absolute conservation laws of charge, baryon number, lepton number, and helicity ("z component of spin"), and our successful handling of the doubled descriptor as baryon–antibaryon quantum numbers at level II, we assume that the quadrupled quantum number represents a baryon–antibaryon pair in conjunction with a lepton–antilepton pair. Then, following the scheme given in Table II, two of the doubled descriptors represent the four spin states obtained by putting together a spin-1/2 baryon with a spin-1/2 antibaryon to form a singlet–triplet system, two of the doubled descriptors correspond to putting together (the same) baryon antibaryon pair to form a singlet–triplet isospin system, and the last two doubled descriptors to the singlet–triplet spin system obtained from spin-1/2 lepton and a spin-1/2 antilepton. Explicitly, the 16 column of equation (2.6) is then $(B,\bar{B},l,\bar{l},s_B^+,s_B^-,s_{\bar{B}}^+,s_{\bar{B}}^-,i_B^+,i_B^-,i_{\bar{B}}^+,i_{\bar{B}}^-,s_l^+,s_l^-,s_{\bar{l}}^+,s_{\bar{l}}^-)$. Anyone familiar with Feynman rules will see immediately that if we interchange rows pairwise when we change the direction of an arrow, we have the usual rule that spins, particle–antiparticle designation, and charge reverse under time reversal, and that we can conserve quantum numbers in the same way we did at lower levels.

The physical interpretation of the individual states in the hierarchy is now straightforward. When we have (1111...) we have 16 spin states$\times$4 isospin states or 64 in all. Note that we now have to talk about conservation of the "z component of isospin," which is equivalent to charge conservation in this context. Note also that we are referring to helicity rather than "spin" in a 3-space or 4-space sense. This is not to carry any implications about "rotations" until we have constructed some discrete approximation to "space–time," which we have yet to do. The (0000...) columns are also easy to interpret. Three of them carry isospin without spin, like pions; nine of them carry both spin and isospin, like the three spin $\times$ three charge states of the ρ mesons; three have isospin zero in three spin states like the ω meson. All of these 15 mesons come from rows associated with the baryons. The remaining three spin states associated with the leptons we identify with the two helicity states of the photon (γ) and the Coulomb field. The mesons can be put together with the γ to form $3\times15=45$ states. Thus the total number of states is $64+15+3+3\times15=127$ as required.

The presence of the γ is particularly interesting. Empirically the mesons all have finite mass and dimensionally speaking explore "distances" of the order of 10^{-13} cm, where many people agree our ordinary ideas of space–time are suspect. But the γ, being massless, has effects of infinite range, and in the large-number limit goes over, via the correspondence principle, to the classical electromagnetic field. One might think that, prior to the development of an explicit dynamics, we should not be able to get quantitative results from the theory at this stage. But the presence of the γ and the Coulomb field in our interpretation allows us to discuss, at least heuristically, a remarkable calculation.

The calculation was originally achieved by A. F. Parker-Rhodes (1978), who justifies his physical interpretation of the hierarchy, and of more extended structures, on the basis of his theory of indistinguishables. Unfortunately, this theory requires considerable logical development for consistent presentation since objects that can be counted as two when together, but that are truly indistinguishable when separate (called "twins"), cannot be grouped in ordered sets; they can, however, be grouped in such a way as to define a unique cardinal for the group or "sort." Thus a "sort theory" dealing with this possibility has to be developed, based on the three parity relations "identical," "distinguishable," and "twins"—together with their negations. This requires a semantic theory, using two-valued logic, for discussion of the object theory, and an implication language, again using two-valued logic, for the statement and proof of theorems. However important the theory of indistinguishables may be, Parker-Rhodes' ideas of interpretation are inconsistent with those developed in this paper, and we give his deductions in an amended form. We expect that before very long a consistent presentation on our own principles will have been reached, but the form we give below is to some extent a compromise with conventional thinking. Our excuse for (in a sense) premature publication is the astonishing accuracy of the result. We believe that the presentation we give here is believable in terms that are closer to ordinary quantum mechanical usage—once one is willing to make the conceptual leap that allows the discussion of quantum ideas *prior* to any mention of space–time.

We have seen that the three stable levels of the hierarchy can be viewed as systems carrying the quantum numbers of baryon–antibaryon pairs and lepton–antilepton pairs and the associated bosons. Since comparison between any two such systems leads to a third, and all three levels map up or down, it seems appropriate to think of the hierarchy as containing all 137 possibilities with equal *a priori* probability. But to discover the actual structure, we must somehow "break into" this closed system, which necessarily requires a column that is not one of themembers of the hierarchy. The example we pick is the electron.

Using the specific choice of row designations already introduced, i.e., $(B,\bar{B},l,\bar{l},s_B^+,s_B^-,s_{\bar{B}}^+,s_{\bar{B}}^-,i_B^+,i_B^-,i_{\bar{B}}^+,i_{\bar{B}}^-,s_l^+,s_l^-,s_{\bar{l}}^+,s_{\bar{l}}^-)$, an electron with "spin up" is (0010 0000 0101 1000) and with "spin down" is (0010 0000 0101 0100).

In order to couple this column into the hierarchy, we have to introduce some new sort of vertex that does conserve quantum numbers; just how does not have to be specified for our current purpose. Presumably this can be done in the same way that we introduced an ordered meson–baryon vertex at level II. The only member of the 137 columns in the hierarchy that does not change the electron spin or charge, or refer to irrelevant quantum numbers, is the Coulomb case. So we assume that the electron couples to this with a probability of 1/137. This member of the hierarchy then communicates with all the others in a random fashion, eventually ending up again with the Coulomb case and back to the electron. In this respect we view the hierarchy as resembling something like the "vacuum fluctuations" of quantum field theory. The reason that this can lead to a result is that the electron cannot coincide with those members of the hierarchy that contain electron–positron pairs while this process is taking place, thanks to the exclusion principle. Particularly since we have as yet not made use of the exclusion principle, an assumption more in keeping with our basic statistical approach (which has the same effect on the calculation) is that the statistical uncertainties in the concept of "length" at nuclear dimensions do not allow us to discuss Coulomb energy separations for lengths smaller than some distance d. Thus the process necessarily involves some space–time separation or interval between the electron and the hierarchy, which we will estimate statistically. Further, since we have no reference frame to refer this distance to, the resulting charge distribution relative to this space–time interval must also be distributed statistically, subject only to charge conservation. The calculation we present is of the ratio of the square of this statistically smeared-out charge to the statistically estimated distance of separation, equated, as is often assumed, to the electron rest energy $m_e c^2$. Schematically, the process we are computing is shown in Figure 4.

Our first step is to take out the dimensional factors and thus reduce the statistical part of the calculation to dimensionless form. The square of the charge is e^2; it is smeared out into two (or more) parts over some

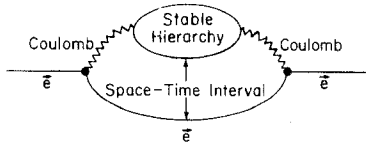

Fig. 4. Schematic representation of the electron self-energy.

distance r. We introduce a random variable x to represent the charge in one part, and, in order to conserve charge between two parts, write the square of the charge as $e^2x(1-x)$. As we have already argued, the coupling we should use at this stage in the development of the theory is $1/137$, not the empirical value of the fine-structure constant α, so $e^2=\hbar c/137$.

Because of the statistical uncertainty in the concept of length at nuclear dimensions, or because of the exclusion principle, there is some distance of closest approach d, which acts as a cutoff in the distance r. Since the only stable mass other than the m_e we are computing is the proton mass m_p, and proton–antiproton pairs occur in the levels of the hierarchy, it seems reasonable to take this shortest distance we can define to be the Compton wavelength of a proton–antiproton pair $d=h/2m_pc$; our second random variable y is then defined by $r=yd$, with $y \geqslant 1$. We like the idea of introducing Planck's constant into the theory as a basic measure of the breakdown of the concept of macroscopic length.

The random variable x represents the charge in a system with three degrees of freedom smeared out statistically and interacting with the remaining charge $1-x$. If we could cut the charge into two pieces, like a hunk of butter, x would vary between 0 and 1. But in our interpretation the hierarchy contains pieces with both positive charge $(p\bar{n},\pi^+,\rho^+,\ldots)$ and negative charge $(\bar{p}n,\pi^-,\rho^-,\ldots)$ as well as neutral and internally neutralized systems, all of which communicate with each other in the stabilization process. Hence, if we look at all the possibilities, and maintain overall charge conservation, x can have any value between $-\infty$ and $+\infty$. Once we have gone beyond the first separation, we have no way of knowing whether the Coulomb energy we are evaluating is attractive (unlike charges) or repulsive (like charges) outside of the interval $0<x<1$. Statistically the positive and negative effects outside this interval must cancel. This statement is not obvious, as has been pointed out to us by F. Levin. To explain it, we note first that if Figure 4 represented a single process the charge would have to follow the electron line and there would be no charge smearing. But in fact we are computing a statistical average of such processes in which we assign the charge to two pieces according to the random variable x as ex and $e(1-x)$. The probability of this separation taking place at one vertex is proportional to the dipole $e^2x(1-x)$. Once we have two smeared out charge distributions further smearing will come from the virtual appearance of charged particle-antiparticle pairs; thanks to our diagrammatic rules, charge is conserved at each vertex in such processes and the overall electric neutrality guarantees that for the first distribution the charge remains ex or $e(1-x)$. Thus although this further smearing can lead to regions with any positive or negative value for the charge, these effects cancel outside the interval $0 \leq x \leq 1$. Further, after the initial smearing, the effective squared charge of each piece is e^2x^2 or $e^2(1-x)^2$, a fact we will need

below. As we can see from Figure 4, in order to reform the electron from these smeared out distributions, we need a second vertex. By microscopic time reversal invariance, which is guaranteed by our equal *a-priori* probabilities, the probability of this closure is again proportional to the dipole $e^2x(1-x)$. We conclude that the overall weighting factor $P(x(1-x))$ to be used in computing $[e^2x(1-x)]$ is proportional to $[x(1-x)]^2$, and is to be normed on the interval $0 \leq x \leq 1$.

Putting this together, we see that

$$m_e c^2 = \langle q^2 \rangle \left\langle \frac{1}{r} \right\rangle = \frac{\hbar c}{137} \langle x(1-x) \rangle \frac{2m_p c}{h} \left\langle \frac{1}{y} \right\rangle$$

or

$$\frac{m_p}{m_e} = \frac{137\pi}{\langle x(1-x) \rangle \langle 1/y \rangle} \tag{3.1}$$

To calculate the expectation value of $1/y$ we need some probability weighting factor $P(1/y)$. We have seen above that the hierarchy has three distinct levels with different interpretations, each carrying charge, so we assume that the distribution of charge in the statistical system has three degrees of freedom, each of which brings in its own random $1/y$. Thus we assume $P(1/y) = (1/y) \cdot (1/y) \cdot (1/y)$ and find that

$$\left\langle \frac{1}{y} \right\rangle = \int_1^\infty \left(\frac{1}{y}\right) P\left(\frac{1}{y}\right) \frac{dy}{y^2} \Big/ \int_1^\infty P\left(\frac{1}{y}\right) \frac{dy}{y^2} = \frac{4}{5} \tag{3.2}$$

If the charge splitting x had only one degree of freedom, the expectation value of $x(1-x)$ using the weighting $P(x(1-x)) = x^2(1-x)^2$ would be

$$K_1 = \langle x(1-x) \rangle_1 = \int_0^1 x(1-x) P(x(1-x))\, dx \Big/ \int_0^1 P(x(1-x))\, dx = \frac{3}{14} \tag{3.3}$$

Actually, as already noted, we have three degrees of freedom coming from the three levels of the hierarchy. Once the distribution has separated into x and $1-x$ the effective squared charge of each piece is x^2 or $(1-x)^2$, so we can write the recursion relation

$$K_n = \int_0^1 \left[x^3(1-x)^3 + K_{n-1} x^2 (1-x)^4 \right] \Big/ \int_0^1 x^2(1-x)^2\, dx$$

$$= \int_0^1 \left[x^3(1-x)^3 + K_{n-1} x^4 (1-x)^2 \right] \Big/ \int_0^1 x^2(1-x)^2\, dx$$

$$= \frac{3}{14} + \frac{2}{7} K_{n-1} = \frac{3}{14} \sum_{i=0}^{n-1} \left(\frac{2}{7}\right)^i \tag{3.4}$$

Putting this back into formula, using K_3, because of the three degrees of freedom of the hierarchy, we have

$$\frac{m_p}{m_e} = \frac{137\pi}{\frac{3}{14} \times \left[1 + \left(\frac{2}{7}\right) + \left(\frac{2}{7}\right)^2\right] \times \frac{4}{5}} = 1836.151497\cdots \tag{3.5}$$

as compared with the latest empirical result 1836.15152 ± 0.00070 (Barash-Schmidt et al., 1978).

Clearly, in presenting our calculation in this way, we have leaped ahead of what we are justified in doing as an explicit dynamical calculation. But the calculation illustrates one way in which two algebraic quantities can be introduced into the theory in the form of the square of one divided by the other. The specific interpretation is compelling because of the high quality of the numerical result; the critical integer 3 which enters both the charge distribution and the separation as three degrees of freedom is, we are confident, correctly identified as the three levels of the hierarchy. That we should be able to interpret this calculation within our framework is evident. This fact alone puts us in a strong position.

The quality of the result makes it important to discuss corrections which might destroy it. To begin with, we have used the value 137 for $1/\alpha$ rather than the empirical value. As discussed below, because of coupling to level IV, we can anticipate corrections to $1/\alpha$ of order $1/256^2$, which is of the correct order of magnitude. The second correction we can anticipate is in the cutoff parameter d. Our first estimate is almost certainly approximately correct, but does not account for the fact that electrons in the hierarchy are sometimes present and sometimes absent. Hence, we can anticipate a correction to d of order $m_e/2m_p$ as well as in the calculation of the correction to $1/\alpha$. Thus we anticipate something like the empirical result for $1/\alpha$ and must hope that the correction to d will almost exactly compensate for it in our formula. Looked at this way, the calculation can be viewed as a guide to *how* to construct the dynamics, rather than as a prediction of our theory. It has already proved of great value in setting up the classification scheme given in the last section, and in obtaining the kinematic bootstrap (Noyes, 1979) at level I.

Since the language we use for justifying the calculation when exhibited pictorially as in Figure 4 makes the stable hierarchy look like a photon, we can try to extend this analogy. To begin with, if we look at coupling into the hierarchy through transverse photons, these will flip the spin of the electron. But again, for a specified spin of the electron, this can happen in only 1 of the 137 possible cases, so the coupling constant is the

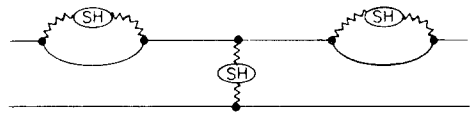

Fig. 5. Single-"photon" exchange between electron and proton.

same as we used in the Coulomb calculation (and including this in our "self-energy" calculation does not alter the result), which is encouraging. So consider an electron and a proton which exchange a "photon" so described. Making the static, nonrelativistic assumption that the mass of the proton does not change with velocity and that its motion does not affect the energy of the system, the additional effect we must consider is that the electron must acquire its own mass both before and after the exchange by the process already considered. This leads to the diagram given in Figure 5.

If the "photon" exchanged in the figure carries any momentum, the diagram cannot represent the whole story, since there will also be the emission of "bremstrahlung" in the final state. So we consider the diagram only for the case when both electron and proton are at rest, but as far apart as we like. This is to be interpreted as an electron and proton bound in the ground state of hydrogen, and contrasted with a free electron and proton with the Coulomb effect shielded out. The second case then is the one already considered except that an inert proton has been added, and the first can be calculated as before, provided we multiply the coupling by the two additional powers of α shown in Figure 5; the statistical calculation remains unaltered. We conclude that the binding energy of the ground state of hydrogen is given by $\alpha^2 m_e c^2 = m_e e^4/\hbar^2$, which is indeed the correct result, in the static case. To obtain the center-of-mass correction we must allow for the motion of the proton, which requires more dynamics than we have developed. Further, to get the excited states, we must be able to describe unstable systems that decay via photon emission, for which we are as yet unready.

We summarize the results of this section by pointing out that we already have in hand the basic ingredients from which atomic theory could be built—stable electrons in correct mass ratio to the proton, photons, and time-ordered Feynman vertices, together with a hint as to how the statistical smearing out of the (not time ordered internally) three levels of the hierarchy's DCsS can stabilize dynamical systems with finite "self-energies." Calculation of genuine dynamical processes such as $\pi^0 \to 2\gamma$ will provide a critical test of whether we are on the right track. We also have the basic ingredients for an approximate nuclear physics—nucleons and

antinucleons coupled to pions and the vector mesons. What is still missing are processes involving beta decay and neutrinos. For these we must go on to level IV.

4. LEVEL IV: WEAK INTERACTIONS AND COSMOLOGY

Returning to the basic mapping matrix from which level IV can be constructed as given in equation (2.6), we see that it will lead, rearranged as a column, to an 18-fold descriptor. The specific mapping actually constructed leads in addition to 126 single descriptors, so it leaves 112 of the 256 available rows unaccounted for. We are confident that mappings that fill all rows can be constructed, and that the multiple descriptor can be 20, 22, ... in other representations, still using our preferred basis. This enormous ambiguity is actually what we expect from elementary particle experiments at high energy. For a while it was thought that there were only three "quark" quantum numbers (up, down, and strange) and two types of leptons (muon and electron with associated neutrinos). But now the "charm" quantum number has been observed, the "upsilon," the heavy lepton called "tau," and most experimental physicists believe that is unlikely to be the end of the story. The important point for us is that none of these new quantum numbers (in contrast to charge, baryon number, lepton number, and helicity) are reliably known to be exactly conserved. Many (e.g., strangeness) are violated in weak decays. But this is what we expect from our combinatorial hierarchy. From our basis of 127 linearly independent columns, we can construct $2^{127}-1$ DCsSs. But only $(256)^2$ linearly independent matrices are available to map, and hence stabilize, them. Thus, even if we happen to start with a particular DCsS, because of our basic statistical assumption that all possible columns are randomly available, after roughly $(256)^2$ discriminations we can expect this set to encounter some column from outside. This will destabilize the system and lead to a "weak decay." Our rough estimate of the coupling constant as $1/(256)^2$ is close to the "weak decay constant" of $10^{-5}m_p$, where, as we saw in our last section, we are constrained to use m_p as our unit of mass. Thus, qualitatively, the scheme predicts weak decays, as was already noted in the first presentation of the hierarchy (Bastin, 1966). We will not even attempt to sketch how this dynamics might work in this paper.

However, we can proceed a little way with the implications of our structure as a classification scheme. The number 18 that occurs in the simplest possible construction of a mapping is already suggestive of quark–antiquark pairs with three colors and three flavors. The 18-fold descriptor then corresponds to conservation of baryon (or quark) number for this system. Doubling 18 of the single descriptors will give us 36 spin

states. We can use $9 \times 18 = 162$ rows to assign color and flavor, leaving $256 - 18 - 36 - 162 = 40$ rows to describe various types of leptons. Working out the details of all this will be a fascinating puzzle. We will certainly want to be intimately in touch with high-energy experiments and current quantum chromodynamics and lepton theories in order to obtain empirical guidance. But, since we can count on destabilization of these "partially conserved quantum numbers," we know that, qualitatively at least, we are dealing with the right structure. We believe it will be more profitable to tackle this problem after we have worked out a firmer hold on atomic and nuclear physics at level III than to plunge into it now.

Before leaving the subject of "weak interactions" we note that we may have been too hasty in placing all of them in level IV. Just as we were able to interpret two columns at level III as a charged lepton (electron) with two spin states, we could leave off the charge descriptors and interpret the resulting columns as an electron-type neutrino. Then, if we can find a way to couple this to the baryons—which we have not yet succeeded in doing —we might be able to include ordinary beta decay at level III. This would not only complete our picture of low-energy nuclear physics, but also could lead to a Weinberg–Salam-type of weak-electromagnetic unification with the same coupling constant of $1/137$. The difficulty will be to show that the coupling to baryons generates a sufficiently large mass for the "W boson" so that the fact that it has yet to be observed experimentally can be accounted for. Then only the more exotic leptons, like the quark quantum numbers, would come in at level IV. We suspect this is the correct route to follow. The check will be whether the extension to level IV gives the quantitatively correct modification of α in accord with experiment. The correction will clearly be of order $1/(256)^2$, which is the right order of magnitude.

But this "high-energy physics" aspect of level IV only deals with the lower levels of its potential complexity—the $2^{127} - 1 \approx 10^{38}$ DCsSs, each of which is a distinct and discriminable entity. Just as we interpret $1/137$ as an approximation to α, we interpret 10^{-38} as an estimate of the gravitational coupling constant between two protons—protons rather than electrons, since we have already accounted for the rest mass of the electron in terms of this unit. At this point a more conventional argument, adapted from a remark of Dyson's (1952), becomes relevant. If we try to count N_e charged particle–antiparticle pairs within a volume whose radius is their compton wavelength, their electrostatic energy is

$$N_e e^2/(\hbar/2mc) = N_e(e^2/\hbar c)2mc^2 \tag{4.1}$$

We interpret this result as saying that if we try to determine the number N_e

for a system with more than 137 pairs by electromagnetic means, we are unable to do so because the energy has become so large that additional pairs could be present, and the counting breaks down. Hence, $N_e = 137$ is the maximum meaningful number of charged particle pairs we can discuss electromagnetically in such a volume (Noyes, 1974).

Extending the argument to gravitation, we see that, since

$$N_G G m_p^2/(\hbar/m_p c) = N_G\left(G m_p^2/\hbar c\right) m_p c^2 \tag{4.2}$$

the maximum number of gravitating protons we can discuss within the compton wavelength of any one of them is $N_G \cong 10^{38}$. In this case, the gravitational field at the surface is so intense that light cannot escape, so this system forms a Laplacian "black hole" (Laplace, 1795). Hence, just as failure of the "fourth level" of the hierarchy to possess linearly independent mappings gives us an estimate of instability to weak decay, the upper limit $2^{127} - 1 \approx 10^{38}$ represents a gravitational instability for systems with large numbers of particles.

Since we have $\sim 10^{38}$ discriminate entities in the scheme, we are logically justified in starting our discussion with the $(10^{38})^2$ possible discriminations between them. For stability, these systems should contain lepton number and baryon number $(10^{38})^2$, although we cannot as yet prove such a conjecture. Given it, the initial discriminations will create all sorts of ephemeral forms of the type already discussed, and a historical system of loci that provides an initial space–time mesh. Once the decays and scattering have proceeded a while, these will settle down to protons, electrons, photons, hydrogen atoms,... and we have started the "big bang." The radiation soon breaks away from the matter, and provides a unique discrete approximation to a space–time framework, locally defined in terms of the cosmic background radiation. Since this "black body spectrum" can be measured locally, it provides us both a cosmic time scale from the temperature, and an absolute frame for measuring particle velocities. Our hope is that we can use this idea to define space–time frameworks more easily connected to laboratory observation than abstract definitions. In particular, since our W boson–photon coupling is discrete, and defined at proto-space–time loci, we should be able to use our dynamic scheme to explain what we mean by a local discrete coordinate system for physical measurement. Only when this task is complete can we tackle the question of what we might mean by a "wave function," and how we are to relate our particular formalism to the successful results obtained by conventional quantum mechanics.

5. CONCLUSION

In this paper we have sketched a physical interpretation of the combinatorial hierarchy, which, if the program can be carried through, should provide a finitist conceptual frame for that fundamental revision of physics which we seek. Our philosophical reasons for adapting this approach are discussed in detail in the opening section. Here we stress that the contact with experiment already established in this paper, together with the indications of structural contact with the classification schemes used in elementary particle physics, and conceptual contact with the fundamental ideas underlying current cosmology, make it clear that no field of physics need be omitted in this synthesis. The original coincidence between the cardinals of the hierarchy and the inverse boson field coupling constants allows us to believe that we have indeed unified strong, electromagnetic and gravitational phenomena in one framework. The weak decay instability is also indicated. Our proposed classification scheme brings in the absolute conservation laws at the correct level, and points toward a weak-electromagnetic unification at that or the next level. Structural contact exists between SU_2, SU_3, and SU_6 (quark) classifications, including an appropriate three-color–three-flavor option flexible enough to allow for new flavors and new heavy leptons. The cosmology should yield the conserved quantum numbers of the universe, some sort of "big bang," and hence the cosmic background radiation as a unique reference system. Since this background is not time reversal invariant, it might even lead ultimately to the explanation of the $K_L - K_S$ decay. So far as we see, no major area of physics has been omitted as potentially outside the reach of a scheme of this structure.

APPENDIX: MATHEMATICAL STRUCTURE OF THE HIERARCHY

This appendix contains a short formal account of the essential mathematical features of a "discrimination system." Throughout, S will denote a nonempty finite set, and $C = \{0,1\}$ will denote either the cyclic group of order 2 (with addition mod 2, or equivalently, Boolean Exclusive-Or as group operation) or the field of two elements (with addition as before, and integer multiplication, or equivalently, Boolean And as field multiplication operation); the context makes clear which usage is intended. The empty set is $\varnothing$; $\mathbb{N} = \{1,2,3,\ldots\}$, $\mathbb{N}_0 = \{0,1,2,3,\ldots\}$; $|X|$ is the cardinality of a set X.

A.1. Discrimination System

A.1.1. Definition. A *discrimination system of type N* ($N \in \mathbb{N}$) is a group S isomorphic to the Abelian group $C^N = C \oplus \cdots \oplus C$, direct sum of N copies of C, together with additional structure as detailed later; its order is $|S| = 2^N$. Thus $x \in S$ iff $x = (x_1, \ldots, x_N)$ $(x_i = 0, 1)$, the group operation (written $+$) on S is termwise addition mod2 (or equivalently, Boolean Exclusive-Or applied to strings of length N), and the group neutral is $e = (0, \ldots, 0)$.

A.1.2. Fact. Besides being symmetric (Abelian) and associative, the group operation $+$ on S is also *discriminative*:

$$(\forall x, y, z \in S) \qquad x + x = e \quad \text{and} \quad y \neq z \Rightarrow y + z \neq e$$

i.e., $+$ can "discriminate" between a pair of equal elements and a pair of unequal elements; the (unique) group neutral e is called the "(discrimination) neutral for S."

A.1.3. Theorem. Any set S equipped with a binary operation that is symmetric, associative, and discriminative (with respect to a unique discrimination neutral e) is isomorphic to an Abelian group C^N for some $N \in \mathbb{N}$; the discrimination neutral e is then (identified with) the neutral element in the group C^N.

A.1.4. Remarks. (1) The Boolean dual $u = (1, \ldots, 1)$ of the discrimination neutral $e = (0, \ldots, 0)$ in a discrimination system S is referred to as the "antineutral" for S. (2) The group C^N, $+$ can always be given a multiplication $*$ so that $C^N, +, *$ becomes a field F of prime power 2^N isomorphic to the Galois field $GF(2^N)$; its neutral-free part $C^N \setminus \{e\}$ forms a multiplicative cyclic group of order $2^N - 1$ with identity element which can be chosen to be the antineutral $u = (1, \ldots, 1)$.

A.2. Discriminately Closed Subsets

A.2.0. Remark. To avoid repetition, the abbreviation "*d*" or "*d*-" will be used for the words "discrimination, discriminate, discriminately" as appropriate throughout this and later sections.

A.2.1. Definition. Let S be a *d*-system with neutral e and let $T \subseteq S$ be a subset. Then T is a *dc-subset* (discriminately closed subset) (alias, subset T is *dc*) iff (a) T is *neutral-free* (i.e., $T \subseteq S \setminus \{e\}$), (b) the *e-join* $T \cup \{e\}$ of T is a subgroup in S.

A.2.2. Facts. (1) Conditions (i) and (ii) are equivalent: (i) T is a *dc*-subset of S; (ii) $(\forall x, y \in T)$ $x \neq y \Leftrightarrow x + y \in T$. (2) $S \setminus \{e\}$ is a *dc*-subset

of S. (3) $\varnothing$ is a *dc*-subset of S. (4) Singleton $\{x\}$ is a *dc*-subset of $S \Leftrightarrow x \neq e$. (5) T is a subgroup of $S \Leftrightarrow T \setminus \{e\}$ is a *dc*-subset of S. (6) $R \cap T$ is a *dc*-subset of S whenever R and T are *dc*-subsets of S.

A.2.3. Definition. The *d-closure* T^{dc} of a neutral-free subset $T \subseteq S$ is the smallest *dc*-subset of S containing T; T^{dc} is *d-generated* by T. The *d-union* $R \cup^{dc} T$ of two neutral-free subsets $R, T \subseteq S$ is the *dc*-subset $(R \cup T)^{dc}$. The latter definition extends in an obvious way to families $(T_i)_{i \in I}$ of neutral-free subsets of S.

A.2.4. Facts. Let R, T be *dc*-subsets of S; then (1) T is $dc \Leftrightarrow T = T^{dc}$. (2) $(T^{dc})^{dc} = T^{dc}$, hence T^{dc} is *dc*. (3) $R \subseteq T \Rightarrow R^{dc} \subseteq T^{dc}$. (4) $R^{dc} \cup T^{dc} \subseteq (R \cup T)^{dc}$. (5) $(R^{dc} \cup T^{dc})^{dc} \subseteq (R \cup T)^{dc}$. (6) $R^{dc} \cup^{dc} T^{dc} = (R^{dc} \cup T^{dc})^{dc} = (R \cup T)^{dc} = R \cup^{dc} T$.

A.2.5. Fact. If $\langle T \rangle$ denotes the subgroup generated in S by a subset $T \subseteq S$ and T is neutral-free then $T^{dc} = \langle T \rangle \setminus \{e\}$ and $\langle T \rangle = T^{dc} \cup \{e\}$.

A.2.6. Definition. A subset T of a *d*-system S is a *d-subsystem* of S iff T is a subgroup of the group S and $T \neq \{e\}$.

A.2.7. Fact. If T is a *d*-subsystem of a *d*-system S of type N then T is isomorphic to a nontrivial subgroup $C^M \subseteq C^N$ with $1 \leqslant M \leqslant N$, and $1 < |T| = 2^M \leqslant |S| = 2^N$, and the neutral elements of T and S coincide.

A.2.8. Definition. The *d-complement* of a *dc*-subset T in a *d*-system S is the unique *dc*-subset R such that the subgroup $R \cup \{e\}$ is the direct complement of the subgroup $T \cup \{e\}$ in the group S.

Notation: $R = S \ominus^{dc} T$ and $S = R \oplus^{dc} T$.

Remark. Since $R = S \ominus^{dc} T \Leftrightarrow T = S \ominus^{dc} R$, so s is said to be *d-decomposed* by the *d-complementary* subsets R and T.

A.2.9. Fact. R and T are *d*-complements in $S \Leftrightarrow R \cap T = \varnothing$ and $R \cup^{dc} T = S \setminus \{e\}$.

A.2.10. Definition. Subset T is *d-independent* in S iff T is neutral-free and $(\forall t \in T)\ \{t\}^{dc} \cap (T \setminus \{t\})^{dc} = \varnothing$; otherwise T is *d-dependent* in S.

Remark. A neutral-free subset T is *d*-independent in *d*-system $S \Leftrightarrow T$ is an independent subset in the group S. The definition extends in an obvious way to families $(T_i)_{i \in I}$ of neutral-free subsets of S, thus $(T_i)_{i \in I}$ is a *d*-independent family iff $(\forall k \in I)\ T_k^{dc} \cap [\cup_{i \in I, i \neq k} T_i]^{dc} = \varnothing$.

A.2.11. Fact. These four conditions are equivalent: (a) Neutral-free family $(T_i)_{i \in I}$ is *d*-independent in *d*-system S. (b) Subgroup family $(\langle T_i \rangle)_{i \in I}$ is independent in group S. (c) Subgroup $\langle \cup_{i \in I} T_i \rangle =$

group direct sum $\oplus_{i\in I}\langle T_i\rangle$. (d) Each $x\in\cup_{i\in I}^{dc}T_i$ has a unique representation $x=\oplus_{i\in I}x_i$ with each component $x_i\in T_i^{dc}\cup\{e\}=\langle T_i\rangle$ and $x_j\neq e$ for at least one index $j\in I$.

A.3. Discriminate Morphisms

A.3.1. Definition. A morphism $f: S\to T$ between two d-systems S and T is a *d-morphism* iff f is an injective group homomorphism between the groups S and T. (Thus every automorphism $f: S\to S$ is a d-morphism on the d-system S.)

A.3.2. Remark. An automorphism $f: S\to S$ on a d-system S leaves fixed the neutral e $[f(e)=e]$ and permutes some or all of the members of the maximal neutral-free subset $S\setminus\{e\}$ in S.

A.3.3. Facts. Let $f: S\to T$ be a d-morphism; then (1) $R\subseteq S\Rightarrow f(\langle R\rangle)=\langle f(r)\rangle$; (2) $R\subseteq S\setminus\{e\}\Rightarrow f(R^{dc})=[f(r)]^{dc}$; (3) $R=R^{dc}\Leftrightarrow f(R)=f(R)^{dc}$; (4) $(S_i)_{i\in I}$ is independent (or, respectively, d-independent) in $S\Leftrightarrow[f(S_i)]_{i\in I}$ is independent (or, respectively, d-independent) in T.

A.3.4. Remark. Recall the following: Let G be a group with neutral e; let $E(G)$ be the group of endomorphisms of G with respect to the composition operation $\circ$, $[(f\circ h)(g)=f(h(g))\ \forall g\in G]$; let $A(G)\subset E(G)$ be the subgroup of automorphisms of G. Introducing a second group operation $(+)$ on $E(G)$, namely, pointwise addition $[(f+h)(g)=f(g)+h(g)\ \forall g\in G]$, makes $E(G), +$ Abelian when G is Abelian, and $E(G), +, \circ$ becomes a ring with neutral endomorphism $\mathbf{e}(g)=e$ $(\forall g\in G)$ and identity automorphism $\mathbf{u}(g)=g$ $(\forall g\in G)$. The case where G is a d-system is of special interest.

A.3.5. Theorem. Let S be a d-system of type N with neutral e. Let $E(S)$ be the Abelian group of endomorphisms of S under pointwise addition, with neutral endomorphism $\mathbf{e}$. Then $E(S)$ is a d-system of type $M=N^2$ and order $2^{(N^2)}$, with neutral $\mathbf{e}$ and antineutral $\mathbf{u}$. [Hence pointwise addition is discriminative on $E(S)$.]

A.3.6. Theorem. The ring $E(S), +, \circ$ of endomorphisms of a d-system S of type N is isomorphic to the ring of square $N\times N$-matrices over C.

A.3.7. Theorem. Let $A(S)\subset E(S)$ be the subset of automorphisms of S; then $A(S)\subseteq E(S)\setminus\{\mathbf{e}\}$, i.e., is a neutral-free subset in the d-system $E(S)$, but is not a dc-subset in $E(S)$. Indeed, $A(S)^{dc}=E(S)\setminus\{\mathbf{e}\}$.

> *Corollary.* Every nonneutral endomorphism of S is a finite sum of distinct automorphisms of S.

A.3.8. Remarks. Let S be a d-system of type $N=t$; then $|S|=2^t$, $|E(S)|=2^{(2^t)}$, $|A(S)|=(2^t-1)(2^t-2)(2^t-4)\cdots(2^t-2^{t-1})$; let $r_t=|A(S)|/|E(S)|$, then $r_1=0.5$ and r_t decreases monotonically, and $\lim_{t\to\infty} r_t=0.288788\ldots$ (i.e., for large t about 29% of all endomorphisms on S are automorphisms). For example:

t	1	2	3	4	5	16
$\|S\|$	2	4	8	16	32	65,536
$\|A(S)\|$	1	6	168	20,160	9,999,360	$3.34\ldots(10^{76})$
$\|E(S)\|$	2	16	512	65,536	33,554,432	$2^{256}=1.1579\ldots(10^{77})$

A.3.9. Remark. Recall the following: If $\mathbf{a}\in A(S)$ then we may define an equivalence relation $[\mathbf{a}]$ on S by $x[\mathbf{a}]y\Leftrightarrow x=\mathbf{a}^k y$ for some $k\in\mathbb{N}_0$. Let $s\in\mathbb{N}$ be the least integer such that $\mathbf{a}^s=\mathbf{u}$ (the identity automorphism on S). Then an equivalence class mod$[\mathbf{a}]$ is called an $\mathbf{a}$-cycle of size r where r is its cardinality. Each $\mathbf{a}$-cycle is an orbit of the subgroup $\langle\mathbf{a}\rangle=\{\mathbf{u},\mathbf{a},\mathbf{a}^2,\ldots,\mathbf{a}^{s-1}\}$ in $A(S),\circ$, and vice versa. Its size $r=\min\{0<k\leqslant s|x=\mathbf{a}^k x\}=\text{index}|\langle\mathbf{a}\rangle:A_x|$ where A_x is the stabilizer subgroup $\{\mathbf{b}\in A(S)|x=\mathbf{b}x\}$ of an arbitrary element x in the $\mathbf{a}$-cycle.

A.3.10. Definition. An automorphism $\mathbf{a}\in A(S)$ is *maximal* iff there exists an $\mathbf{a}$-cycle equal to $S\backslash\{e\}$ (i.e., of maximum size $r=2^N-1$); $\mathbf{a}$ is *minimal* iff there exists an $\mathbf{a}$-cycle equal to $\{x\}$ for some $x\neq e$ (i.e., a nonneutral $\mathbf{a}$-cycle of minimum size $r=1$).

A.3.11. Theorem. Each d-system S of type N has (a) at least 2^N-2 distinct maximal automorphisms, and (b) at least 2^N-1 distinct minimal automorphisms.

A.3.12. Remark. The proof of the above theorem requires the construction of some not immediately obvious automorphisms:

(a) Let F be the (Galois) field associated with S [Remark A.1.4(2)], $F_0=F\backslash\{e\}$ its neutral-free multiplicative group, cyclic of order $n=2^N-1$. Choose any one of the $n-1$ generators $b\in F_0$ (thus $b^n=u\neq b$, and $u,b,b^2,\ldots,b^{n-1}$ are all the distinct elements of F_0). Let $\mathbf{a}:S\to S,\mathbf{a}(x)=bx$ (if $x\neq e$), $\mathbf{a}(e)=e$; then $\mathbf{a}\in A(S)$ and for a fixed element $x\in S\backslash\{e\}$ the images $\mathbf{a}^k(x)=b^k x$ $(k=1,\ldots,n)$ are all distinct and exhaust $S\backslash\{e\}$. Thus each such $\mathbf{a}$ is a maximal automorphism on S, one for each of the 2^N-2 generators $b\in F_0$.

(b) Given a dc-subset T in S, let R be its d-complement (Definition A.2.8). Define $f=u'\oplus m':S\to S$ by $f(x)=u'(x)$, $(\forall x\in T\backslash\{e\})$, where u' is the identity automorphism on the d-subsystem $T\backslash\{e\}$, and $f(y)=m'(y)$, $(\forall y\in R\backslash\{e\})$, where m' is a maximal automorphism [by (a) above] on the

d-subsystem $R\setminus\{e\}$. Then $\forall x=t\oplus r\in S=(T\setminus\{e\})\oplus(R\setminus\{e\})$, we have $f(x)=(u'\oplus m')(t\oplus r)=u'(t)+m'(r)$. Thus $f\in A(S)$ and f fixes the dc-subset T. Taking $T=\{x\}$ $(x\neq e)$ makes f minimal, and T is one of its cycles of size 1 (the only nonneutral one).

A.3.13. Definition. The *df-set* (discrimination fixed set) of an automorphism $\mathbf{a}\in A(S)$ is the subset $DF(\mathbf{a})=\{x\in S\setminus\{e\}|x=\mathbf{a}(x)\}$.

Remark. Each $DF(\mathbf{a})$ is a dc-subset; it may be empty [e.g., if $\mathbf{a}(x)=x$ only if $x=e$, i.e., if $\mathbf{a}$ "unfixes" every nonneutral member of S].

A.3.14. Theorem. Each dc-subset T in a d-system S is the *df*-set of some automorphism $\mathbf{a}\in A(S)$.

A.4. Discrimination Hierarchies

A.4.1. Definition. A d-system S of type N *determines* iteratively a sequence $E^0,E^1,E^2,\ldots,E^m,\ldots$ of d-systems where $E^0=S$ and $(\forall m\in\mathbb{N})$ $E^m=E(E^{m-1})$ is the group of endomorphisms of the d-system E^{m-1} under pointwise addition. Then E^m is the *d-system of level m determined by the base d-system* $E^0=S$; it is of type $t(m)$ and order $|E^m|=2^{t(m)}$; A^m denotes the subset of automorphisms in E^m; the neutral (or, respectively, identity) morphisms in E^m are denoted by $\mathbf{e}^m$ (respectively, $\mathbf{u}^m$).

Remark. Where need be, the previous single-level notations such as $S,e,u,E(S),A(S),\mathbf{e},\mathbf{u}$ may now be replaced by $E^0,\mathbf{e}^0,\mathbf{u}^0,E^1,A^1,\mathbf{e}^1,\mathbf{u}^1$.

A.4.2. Fact. Since $t(0)=N$ and $t(m)=t(m-1)^2$, so $t(m)=N^{(2^m)}$ and $|E^m|=2^{t(m)}$ $(\forall m\in\mathbb{N}_0)$.

A.4.3. Remark. Plainly, each E^m is isomorphic to E^0 if E^0 is of type $N=1$. More generally, if $N\geqslant 2$, we have an injective mapping f^m of E^m strictly into E^{m+1} given $(\forall m\in\mathbb{N}_0)$ by $f^m(\mathbf{e}^m)=\mathbf{e}^{m+1}$ and $f^m(x)=\mathbf{a}_x\in E^{m+1}$ $(\forall x\neq\mathbf{e}^m$ in $E^m)$ where $\mathbf{a}_x$ is an automorphism with singleton dc-subset $\{x\}$ as its *df*-set (Definition A.3.13). This mapping can be extended, in certain circumstances, to one that maps many more *df*-sets in one level injectively into an independent set of automorphisms in the next level, in a way now to be made precise.

A.4.4. Construction. Let $N\geqslant 2$; let $m\in\mathbb{N}_0$; let $K^m\subset E^m$ be an independent subset with $|K^m|=k(m)$ $[\leqslant t(m)]$ (Fact A.4.2). Let $K\subseteq K^m$ be any one of the $2^{k(m)}-1$ nonempty subsets of K^m; and define $W_K=\{\mathbf{a}\in A^{m+1}|K^{dc}=DF(\mathbf{a})\}$. (Note that distinct subsets K of K^m have distinct d-closures K^{dc} because K^m is an independent set of elements.) Simple examples with $N=3$, $m=1$, show that we may have $|W_K|>1$. In such a

case, choose precisely one automorphism $\mathbf{a}_K$, say, in W_K and let C^{m+1} be the set of all such choices; thus

$$C^{m+1}=\{\mathbf{a}_K\in A^{m+1}|K\subseteq K^m, K\neq\varnothing, DF(\mathbf{a}_K)=K^{dc}\}$$

[$=C^{m+1}(K^m)$, if we need to refer to the particular set K^m in use]. Hence $C^{m+1}\subset A^{m+1}\subset E^{m+1}$, and $|C^{m+1}|=2^{k(m)}-1$ [$=c(m+1)$, say, by way of definition]. To initialize this construction we define C^0 to be a basis (i.e., a maximal independent subset) for E^0 ($=S$), so that $c(0)=|S|=N$.

A.4.5. Question. Is it possible to construct iteratively the sequence $K^0=C^0, K^1=C^1(K^0),\ldots,K^{m+1}=C^{m+1}(K^m),\ldots$ with each K^m an independent subset (as required by Construction A.4.4)? An obviously necessary condition for this to be possible is this inequality:

$$k(m+1)=c(m+1)=2^{k(m)}-1\leqslant t(m)^2 \qquad [*]$$

since $K^{m+1}\subset E^{m+1}$. A sufficient condition will be given below (A.4.8).

A.4.6. Definition. Let $S=E^0$ be a base d-system of type $N\geqslant 2$.

(A) For $m\in\mathbb{N}_0$, E^m is *d-injectable* into E^{m+1} *via* an independent subset $K^m\subset E^m$ iff there exists at least one choice of automorphisms for the set $C^{m+1}(K^m)$ which makes the latter set independent in E^{m+1}.

(B) A finite sequence $(E^0,E^1,\ldots,E^H)$ of d-systems determined by E^0 is a *d-hierarchy* (*discrimination hierarchy*) of *height* $H+1$ iff these three conditions hold: (1) $H\geqslant 1$. (2) Independent subset K^0 is maximal in E^0. (3) For each $m=0,1,\ldots,H-1$, but not for $m=H$, E^m is d-injectable into E^{m+1} via the independent subset $K^m\subset E^m$, where $K^m=C^m(K^{m-1})$ for each $m=1,\ldots,H-1$.

(C) A d-hierarchy is *trivial* if $H\leqslant 1$, otherwise *nontrivial*.

A.4.7. Remark. The sequence of independent subsets $(K^0,\ldots,K^H)$ in a d-hierarchy $(E^0,\ldots,E^H)$ is a "discriminate spine"; examples show that it need not be unique. If for some m the choice of C^{m+1} is not unique, each possible choice of C^{m+1} gives rise to a different "branch" of the spine.

A.4.8. Main Theorem. A necessary and sufficient condition for the existence of a nontrivial discrimination hierarchy is that the base discrimination system $S=E^0$ be of type $N=2$; and then the discrimination hierarchy is of height $H+1=4$.

A.4.9. Remarks. (1) The necessity follows from the condition [*] in (A.4.5): Since $H\geqslant 2$ and E^m is d-injectable into E^{m+1} for $m=0,1,\ldots,H-1$, condition [*] holds in particular for $m=0,1$. Thus, $k(0)=N$ and $2^N-1\leqslant t(0)^2=N^2$ so that $2\leqslant N\leqslant 4$; and $k(1)=2^{k(0)}-1=2^N-1$, hence

$2^{k(1)}-1=2^{(2^N-1)}-1\leqslant t(1)^2=[t(0)^2]^2=N^4$ so that $N=2$ as asserted. But then [*] can be satisfied for $m=0$, 1 or 2 but not for $m\geqslant 3$; in particular E^3 is not then d-injectable into E^4 so that $2\leqslant H\leqslant 3$; i.e., the discrimination hierarchy must have height 3 or 4. (2) A theoretical proof of the existence of a discrimination hierarchy with $N=2$ and height $H+1=4$ has been provided by C. W. Kilmister (1978) and will be reported elsewhere; an empirical representation using matrices over the field $\{0,1\}$ has been constructed by H. P. Noyes and described in the main body of the paper to which this Appendix is attached.

A.4.10. Remark. The connections between the notations used in this Appendix and those used in the main text (e.g., in Table I) are as follows:

Table I	Appendix
Index of level $=1,2,3,4$	$m=l-1=0,1,2,3$
"Dimension" of level $=n(l)$	"Type" of level $=t(m)=n(l-1)$
Number of independent columns is $j(l)$	Number of independent elements in subset K^m is $k(m)=j(l-1)$
Number of discriminately closed subsets used in level is $d(l)=2^{j(l)}-1$	Number of automorphisms chosen for the subset C^{m+1}, corresponding one-to-one with dc-subsets used in previous level is $c(m+1)=2^{k(m)}-1=d(l)$

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REFERENCES

Amson, J. (1976). "Discrimination Systems," 20 pp. (unpublished).

Barasch-Schmidt, N. et al. (1978). "Particle Properties Data Booklet April 1978," p. 2, from *Physics Letters*, **75B**, 1–250.

Bastin, T. (1966). "On the Origin of the Scale Constants of Physics," *Studia Philosophica Gandensia*, **4**, 77–101.

Bastin, T. (1976a). "A Combinatorial Model for Scattering," Report to the Science Research Council (U.K.), 57 pp. (unpublished).

Bastin, T. (1976b). "An Operational Model for Particle Scattering Using a Discrete Approach," Report to the Conference on "Quantum Theory and the Structures of Time and Space 2," Tutzing, 6 pp. (unpublished).

Dyson, F. J. (1952). "Divergence of Perturbation Theory in Quantum Electrodynamics," *Physical Review*, **85**, 631–632.

Finkelstein, D. (1969). "Space–Time Code," *Physical Review*, **184**, 1261–1271.

Finkelstein, D. (1979). "Holistic Methods," submitted to *International Journal of Theoretical Physics*.

Kilmister, C. W. (1978). private communication.

Laplace, P. S. (1795). *Exposition du Monde*, Vol. II, p. 305, Paris.

Noyes, H. P. (1957). "The Physical Description of Elementary Particles," *American Scientist*, **Vol. 45**, 431-448.

Noyes, H. P. (1974). "Non-Locality in Particle Physics," 55 pp., SLAC-PUB-1405.

Noyes, H. P. (1975). "Fixed Past and Uncertain Future: A Single-time Covariant Quantum Particle Mechanics," *Foundations of Physics*, **5**, 37–43 (Erratum **6**, 125, 1976).

Noyes, H. P. (1976). "A Democritean Phenomenology for Quantum Scattering Theory," *Foundations of Physics*, **6**, 83–100.

Noyes, H. P. (1976). "A Democritean Approach to Elementary Particle Physics," from Proceedings of the Summer Institute on Particle Physics, Martha Zipf, ed., pp. 239–259, SLAC Report No. 198 (issued as a separate document as SLAC-PUB-1956, 1977).

Noyes, H. P. (1978). Private communication.

Noyes, H. P. (1979). "The Lowest Level of the Combinatorial Hierarchy as a Particle Antiparticle Quantum Bootstrap," SLAC-PUB-2277.

Parker-Rhodes, A. F. (1978). "The Theory of Indistinguishables," 208 pp., (unpublished).

Weizsäcker, C. F. von. (1978). "Temporal Logic and a Reconstruction of Quantum Theory," 88 pp., presented at the Conference on "Quantum Theory and the Structures of Time and Space 3," Tutzing (unpublished).

Wheeler, J. A., and Patton, C. M. (1977). "Is Physics Legislated by Cosmogony?" pp. 19–35, in *The Encyclopedia of Ignorance*, R. Duncan and M. Weston-Smith, eds., Pergamon, Oxford.

Whiteman, J. H. M. (1971). "The Phenomenology of Observations and Explanation in Quantum Theory," in *Quantum Theory and Beyond*, T. Bastin, ed., pp. 71–84, Cambridge.